YES, BUT WHY?

Teaching for understanding in mathematics

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Introduction

I was never a keen mathematician as a child. When I first started school, I was good at maths, but I never really understood it. I followed algorithms, and churned out answers that matched those of the teacher, but I was never satisfied with the process. I knew how to check if things were correct, but it was the steps to get there that bothered me. I didn't understand them, and they were left to my imagination to try and explain. Mathematics gradually became a mysterious entity, whose rules and steps I was expected to unquestioningly memorise – which I dutifully did. However, the process of storing numerous algorithms and their quirky properties became increasingly tedious, and I fell out of love with the subject that once intrigued and excited me. It still felt important though. I continued to study it alongside other subjects right through to my graduation from university, and went on to become a teacher of … computing, although, my skillset inevitably brought me back into the maths classroom. I was determined not to teach mathematics the way it had been taught to me. I revisited the various topics on the curriculum with a determination to understand everything. Every detail. Getting the right answers wasn't enough. Where did they come from? What was the point of each step for each solution? I wasn't interested in stating formulae, I was interested in deriving them. I wasn't satisfied with being told there were three types of average (there are more, I was lied to), I wanted to know *why*, I wanted to know who decided upon them as standard measures and I wanted to know about the struggles that people endured to make people listen the first time these ideas were mooted. Who was Pythagoras? Why do so many things have such bizarre names? Surds? Quadratics? Where do these words come from?

I needed to know. I wanted students to know. I wanted them to have the opportunity to genuinely understand, rather than passively accept mathematics.

As a teacher, this knowledge has transformed the way in which I teach. Concepts have origins, stories, logic, connections and intuitiveness – rather than being isolated sorcery. The feeling when a student gets the right answer is incomparable to when they say 'that makes *sense*'.

As a student, this additional understanding is transformational. Answers begin to *look* right, methods can be adapted and applied to different contexts, and students no longer need to rely on the memorisation of hundreds of disparate facts. Each concept is suddenly connected and the sophisticated beauty of mathematics becomes clear. Perhaps even more importantly, mathematics becomes a joyful experience.

This book is intended as a complement to your existing subject knowledge. It is written with an underlying assumption that you are already familiar with many of the algorithms used to solve maths problems – and although time is spent revisiting those procedures, the emphasis here is on *how* they work, *where* mathematical rules come from and *why* they're important.

1

Types of Numbers (Part 1)

It feels sensible enough to start at what is arguably the most intuitive element of mathematics – counting. We can all do it. Furthermore, alongside many other animals, we are instinctively able to do a kind of counting called **subitising**. Subitising is an intuitive sense of comparison of quantity, without a need for symbols. A basic ability to subitise small numbers has allowed us, and everything else in the food chain, to check whether all of our children are still around, or to count the number of predators near our next meal. This ability is apparent very early on too. Children who have not yet learnt to speak, or indeed stand, will still show surprise when one of a small number of identical objects is removed without replacement when they aren't looking.

Counting is slightly different. It is both the ability to order numbers in sequence, but also an ability to total a quantity. Where subitising may allow us to sense which size is bigger, counting allows us to say how many of each thing there is (Figure 1.1).

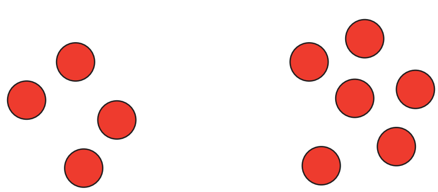

Figure 1.1 Subitising is the ability to instantly recognise small quantities without counting

Counting is the simplest sequence there is, but what we perhaps don't appreciate, is how the English language has conveniently made it easier for us. Our number language is systematic and highly structured. There is a clear pattern in sets of ten that can be picked up quickly by anyone looking at it for the first time. However, the Chinese perhaps have the most straightforward number system:

The Chinese language of number (Table 1.1)

Table 1.1 The Chinese number system is more intuitive than our own

Decimal	Chinese	Literal translation	Decimal	Chinese	Literal translation
1	Yi	One	7	Qi	Seven
2	Er	Two	8	Ba	Eight
3	San	Three	9	Jiu	Nine
4	Si	Four	10	Shi	Ten
5	Wu	Five	11	Shi Yi	Ten One
6	Liu	Six	12	Shi Er	Ten Two

That's pretty clear and sensible. And if we were inventing words for numbers in English all over again I'd probably opt for that structure for clarity. So what about our own language?

The English language of number

At first glance we don't fare too badly:

'one' 'two' 'three' … 'twenty-one' … 'thirty-two' … 'forty-three'.

There is a sound structure in place, allowing us to understand quite easily the quantity being portrayed by the words themselves.

Alas, there are a few peculiarities. One to ten are quite straightforward, but then we get to eleven. What's wrong with eleven? Well for a start, it doesn't crop up again until we count a further hundred numbers forward. In fact, the whole set of numbers between eleven and nineteen are a bit of a pain if you think about it. So why aren't they called 'ten-one', 'ten-two', 'ten-three' and so on? That would at least fit into the pattern of the other ninety numbers before we reach one hundred. Also why does the pattern reset itself after every ten? Why not after every five? Or sixteen?

Why are eleven and twelve such misfits?

Eleven and twelve are arguably the most peculiar, ill-fitting names for numbers in our number system. Their names originate from German. English as a language is largely Germanic, and German, like English, is fairly unique in having the rather awkward 'eleven' (*elf* in German) and 'twelve' (*zwolf*). And if you go back to *Old* German, these words effectively mean 'ten-and-one' (*ein-lif*), and 'ten-and-two' (*zwo-lif*). So once upon a time they *did* fit in with a conventional, sensible structure. Over time, as with most words, they have evolved and changed a bit, and hence feel a little unsuitable in their current format. But it could be worse – we have historically used quite a few peculiar alternative names for numbers such as 'four score and eight' or 'four and twenty'. Spare a thought then for our Welsh friends, who say 'nineteen on twenty' (literally translated) to describe thirty-nine, or the Danes who say 'two-and-a-half times twenty, and four' for fifty-four. Suddenly eleven, twelve and the teens don't seem so bad after all.

Unique descriptions of numbers in English

couple	(2)
pair	(2)
brace	(2)
dozen	(12)
baker's dozen	(13)
score	(20)
gross	(144)
myriad	(10,000)

Teacher tip

Get students to create their own number systems. They can create their own words and symbols. See if they start to create a logical system (or guide them towards doing so). Does their system repeat its structure to mimic our own? Does it repeat its structure in a different way? You could encourage students to try a different base under the guise of an alien number system. Perhaps they could create new symbols that more accurately represent the quantities they represent. For example three might become 'o_o_o'. This kind of activity can help students understand and appreciate our number system and the logical structure it follows.

Why do we count in base-10?

Base-10 is known as the **decimal system**. We have ten digits that we use in organised combinations for all numbers: 0–9. The reason we use base-10 is incredibly likely to be because we have 10 fingers. The Simpsons therefore, should count in base-8, as they have 8 fingers. That would mean going from 1 to 7 and then straight to 10. Which would mean … 8. The mind boggles. Binary and hexadecimal are other examples of number systems in different bases. Binary is base-2 (zeroes and ones), and hexadecimal uses letters (A–F) to continue the symbols past 0–9 to create a base-16 system (Table 1.2). Both are used in computing.

Table 1.2 The numbers 1–16 in base-10, base-2 and base-16

Decimal	Binary	Hexadecimal	Decimal	Binary	Hexadecimal
1	1	1	9	1001	9
2	10	2	10	1010	A
3	11	3	11	1011	B
4	100	4	12	1100	C
5	101	5	13	1101	D
6	110	6	14	1110	E
7	111	7	15	1111	F
8	1000	8	16	10000	10

Teacher tip

To help students understand and appreciate the decimal system (and what alternative systems could look like), get them to tape two fingers together on one or both hands. Role-play the idea that the students are now cartoon characters with a different number system. See if they can accurately count in a new base system.

Odd and even numbers

Once children get familiar with counting, the next step is often to introduce the first example of *categorisation* in our number system – odd and even numbers.

Odd numbers (a number of **odd parity**) always end in a 1, 3, 5, 7 or a 9. Contrastingly, even numbers (a number of **even parity**) end in a 2, 4, 6, 8 or 0. However, these facts alone do not make a particularly good definition. For example, 3 is odd, but would 3.0 be even? Of course not. What about 3.2? Is that even or odd? Let's look a little deeper.

Why do we categorise odd and even numbers?

To answer this we must first appreciate two things – the usefulness of categorisation in mathematics, and the properties of odd and even numbers. Mathematicians have a love of categorising. If a set of numbers has a special property, then categorising helps us identify, describe and study them. Odd and even numbers are perhaps the simplest example of categorisation in maths. There are also several other types of numbers that students are exposed to at school, such as prime numbers, square numbers and negative numbers. Each of these is categorised due to its own unique properties. We'll look at them later.

What are the properties of odd and even numbers?

Even numbers can be represented pictorially (Figure 1.2).

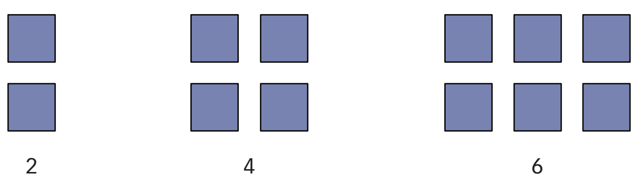

Figure 1.2 Even numbers pictorially are even ended

As you can see, they are always nicely *even* ended.

Odds however …

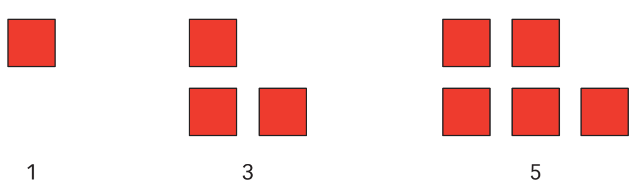

Figure 1.3 Odd numbers pictorially are uneven

have an awkward (some might say … odd) bump (Figure 1.3).

By drawing odd and even numbers in this way, we can answer some other puzzling questions quite easily.

Why do two even numbers always sum to make another even number?

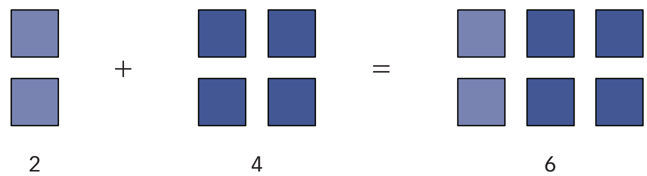

Figure 1.4 The addition of two even numbers sums to an even number

You can see from Figure 1.4 that any two even numbers added together would always result in a larger, but still rectangular shape, which is even.

Why do two odd numbers always add to make an even number?

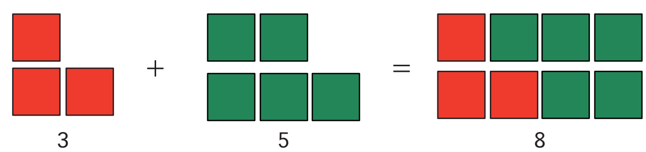

Figure 1.5 The addition of two odd numbers sums to an even number

This time, we can see that two odd shapes equal an even (rectangular) shape, by rotating one of the odd shapes 180° (Figure 1.5).

Why do an odd and an even number always add to make an odd number?

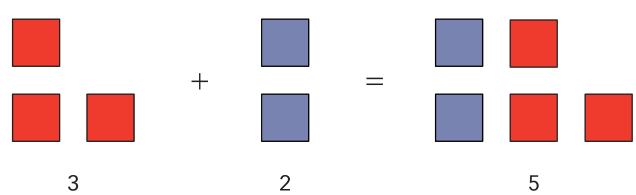

Figure 1.6 The addition of an odd and an even number sums to an odd number

Here we have our only odd answer when adding two numbers. You can see how an odd and even shape will never yield an even rectangle when adding (Figure 1.6).

What are the multiplicative properties of odd and even numbers?

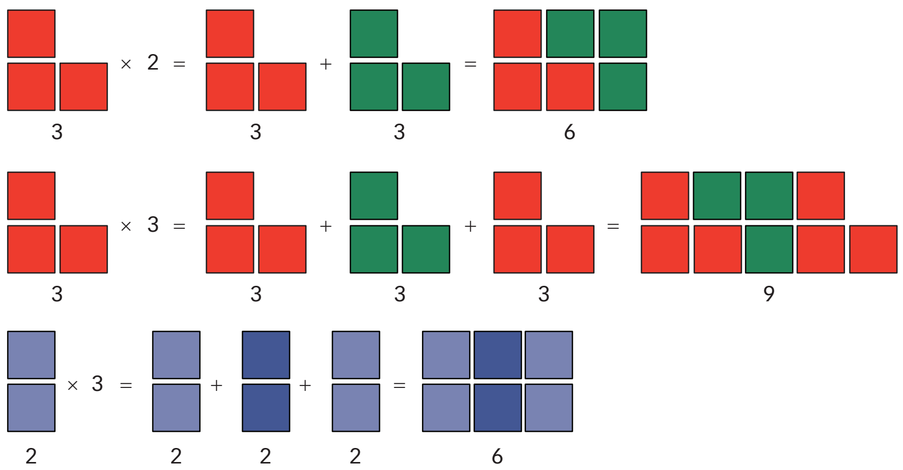

Figure 1.7 The multiplicative properties of odd and even numbers

Again, with careful observation of the shape of even and odd numbers, you can see in Figure 1.7 how only an odd number multiplied by another odd number will produce an odd answer.

We're getting closer to a nice mathematical definition of odd or even numbers, but first let us consider fractions and decimals, which incidentally are neither odd nor even.

Why are fractions and decimals neither odd nor even?

While it is quite understandable why one might assume that *anything* ending in an odd number is odd, or ending in an even number is even, the idea breaks down the more you analyse it.

Let us start with fractions. Would we assume that if the numerator is odd, then the fraction is odd? If we did, then $\frac{3}{5}$ would be odd. But $\frac{3}{5}$ is equivalent to 0.6, which ends in an even number, so it must be even, mustn't it? Furthermore, $\frac{3}{5}$ is equivalent to $\frac{6}{10}$. So now would it be considered even? Quite simply, the odd/even argument doesn't work when we use fractions.

Now let us consider decimals. Assuming a number ending in an even digit is even, and a number ending in an odd digit is odd, then 0.3 would be odd. However, 0.3 is the same as 0.30, which would be even. Not convinced? Here's another problem: recall the rules we established earlier about odd numbers added to odd numbers (they always make an even number). Now consider:

$$1.5 + 1.5 = 3$$

Well, either our rules are ruined, or decimals shouldn't be in the club.

Let us conclude then, with an appropriate definition of odd and even numbers:

An even number is an integer of the form $n = 2k$, *where k is also an integer*. Use Table 1.3 if you need help visualising the algebra.

Table 1.3

Value for n	Value for k	$n = 2k$
0	$k = 0$	$0 = 2 \times 0$
2	$k = 1$	$2 = 2 \times 1$
4	$k = 2$	$4 = 2 \times 2$
6	$k = 3$	$6 = 2 \times 3$

An odd number therefore, is simply any integer that is *not* even. That is, an integer of the form $n = 2k + 1$, where k is (still) an integer (Table 1.4).

Table 1.4

Value for n	Value for k	$n = 2k + 1$
1	$k = 0$	$1 = (2 \times 0) + 1$
3	$k = 1$	$3 = (2 \times 1) + 1$
5	$k = 2$	$5 = (2 \times 2) + 1$
7	$k = 3$	$7 = (2 \times 3) + 1$

Since we're throwing in terms like **integer** now, we should probably visit some of the other broad categories in our number system (Table 1.5, Figure 1.8).

Table 1.5 Symbols for types of numbers and their meanings

Category	Description
Natural numbers, $\mathbb{N}$	Natural numbers are all of the **positive whole numbers**. 1, 2, 3, … They are sometimes referred to as 'counting numbers', and, depending on your definition, can be considered either exclusive or inclusive of the number zero (there is no universal agreement on whether it should be included, as it is 'non-negative' so loosely fits with some definitions)
Integers, $\mathbb{Z}$ (The Z stands for ***Zahlen*** meaning 'numbers' in German)	Integers are all the **whole numbers**. Integers are inclusive of negative numbers *and* zero
Rational numbers, $\mathbb{Q}$ (the Q stands for **quotient**, meaning the result when dividing one quantity by another)	In our ever expanding net, rational numbers are inclusive of **integers *and* fractions**
Irrational numbers $\mathbb{R}\backslash\mathbb{Q}$ (meaning real numbers excluding rational numbers)	Irrational numbers *cannot* be expressed as fractions, π being a famous example
Real numbers, $\mathbb{R}$	This is the net that catches all of the above. Real numbers includes everything apart from imaginary and complex numbers, but let's not worry about those here

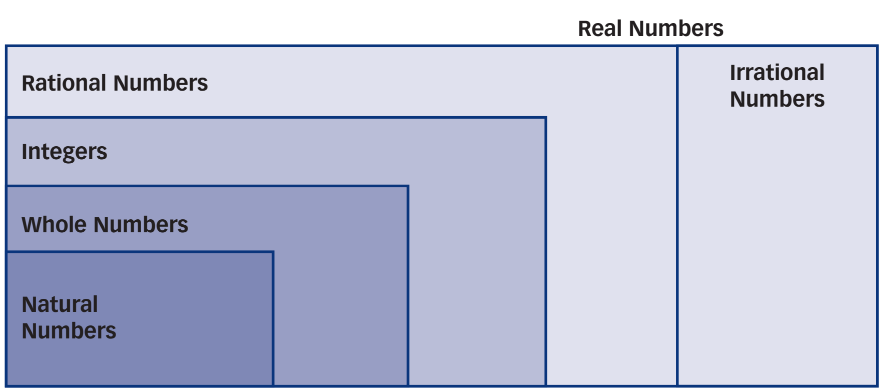

Figure 1.8 Broad categories of our number system

Zero (part 1)

Zero is one of the younger numbers in the number system. It has a long and troublesome history as a bit of a rebel but was a significant breakthrough in the world of maths. Put simply, zero changed the way we use mathematics forever.

Why did zero arrive so late?

Zero simply wasn't a consideration for most cultures in ancient times. There was no real need for a specific number or symbol to explain *the absence* of something. For example, 'There are no cows in this field' does not require the number zero to convey understanding. Before zero, the typical structure of numerical symbolism was to use new symbols or symbol repetition as you ascended the number line. Roman numerals are a great example. X is ten, L is 50, etc. Romans used a logical structure to their number system, which was based around seven key symbols: I (1), V (5), X (10), L (50), C (100), D (500) and M (1000). Each number could be represented using a combination of those symbols, in descending order from left to right (occasionally with subtraction to avoid writing the same symbol several times, e.g. 4 is IV – 'one less than five', indicated by having a smaller number in front of

Table 1.6 Numbers and their Roman numeral equivalents

Decimal number	Roman numeral	Decimal number	Roman numeral	Decimal number	Roman numeral
1	I	51	LI	91	XCI
2	II	52	LII	92	XCII
3	III	53	LIII	93	XCIII
4	IV	54	LIV	94	XCIV
5	V	55	LV	95	XCV
6	VI	56	LVI	96	XCVI
7	VII	57	LVII	97	XCVII
8	VIII	58	LVIII	98	XCVIII
9	IX	59	LIX	99	XCIX
10	X	60	LX	100	C

a larger one (Table 1.6). Strangely, this convention didn't quite follow into modern clock faces, which often have 4 written as IIII instead of IV).

A major downside to doing things this way around, is that eventually you begin to use ridiculously long combinations of symbols for relatively small numbers.

For example: LXXXVIII is 88, and MCMLXXXVII is 1987. Worse still, imagine how addition would work in the Roman numeral system. Mental addition would be fine for small numbers, but to work out bigger sums would take a lot of tedious work, cross-referencing and symbol exchanging. In fact, Romans used to use a combination of an abacus and their Roman numeral system to essentially convert from one to the other, then back again.

What is strange, is that the idea of a placeholder (which eventually became 'zero') arose *before* Roman numerals, but was not used by the Romans. Evidence of the conceptual beginnings of zero can be found on ancient Babylonian tablets circa 2000 BC where empty spaces were used to indicate place value. So the difference between 1, 101 and 1001 would be the size of the gap between the 1's. However, this is a rather poor system, as different people could interpret gaps differently, and only context could help someone differentiate between 1 and 100.

So it was a bit of a stroke of genius when someone started using a *symbol* to represent this gap, and it came from a fairly unlikely source. Some 5000 years ago, traders in Mesopotamia began using double dashes, like //, to mean 'nothing goes here' when writing out large numbers. Who needs famous mathematicians? The use of the actual '0' symbol developed in India, where some of the first written uses of zero can still be seen today. India was the first country to truly conceptualise the idea of zero and see the benefits of it as part of the number system.

This breakthrough suddenly allowed people to write 101 as '1 // 1' instead of 1 1. The brilliant consequence of this, is that suddenly a mere nine numerical symbols, plus some dashes (so … ten symbols) are required to represent any whole number. Pretty handy, especially considering that up to this point, deciphering the difference between 11, 101 or 1001 was surprisingly tricky.

Place value

The adoption of zero, or any other symbol acting as a zero, allowed for a coherent and effective place-value system. But what *is* place value? Well, it is the notion that any digit in a number has a specific quantity attached to it, and that quantity differs depending on *where* the digit resides.

Consider 2, 20 and 200. Each number contains the digit 2, but the value of 2 means something quite different in each example.

2 = two units

20 = two tens and zero units

200 = two hundreds, zero tens and zero units.

You can see how zero is playing its part beautifully. It is creating a specific place for the 2 to go each time. Younger students often struggle with aligning these digits. As such, it is common to provide columns as in Figure 1.9.

Hundreds	Tens	Units		Tenths
		0	.	2
		2	.	0
	2	0	.	0
2	0	0	.	0

Figure 1.9 Digits aligned in columns

Our base-10 system ensures that each place in a number is an increase in value of a power of ten.

Hence:

$$20 = 2 \times 10 = 2 \times 10^1$$

$$200 = 2 \times 10 \times 10 = 2 \times 10^2$$

$$2000 = 2 \times 10 \times 10 \times 10 = 2 \times 10^3$$

$$\dots$$

$$3480 = (3 \times 10 \times 10 \times 10) + (4 \times 10 \times 10) + (8 \times 10) + 0$$

Teacher tip

When looking at this for the first time, students may well struggle to quantify or visualise larger numbers. Manipulatives such as place-value blocks are crucial here to help students relate the abstract ('10' for example) with the concrete (say, Cuisenaire rods).

Not only do these help quantify numbers, they also help students understand that 20 units is equivalent to two tens, five tens is half of 100, etc.

These powers can continue infinitely, but place value also allows us to write numbers *between* integers with **decimal places**.

Decimal places

The use of a decimal point '.' differentiates whole number values from decimal values (confusingly, it is sometimes written as a comma ',' rather than a dot). As before, these ascend in powers of ten (ten hundredths are equivalent to one tenth, etc.) (Figure 1.10).

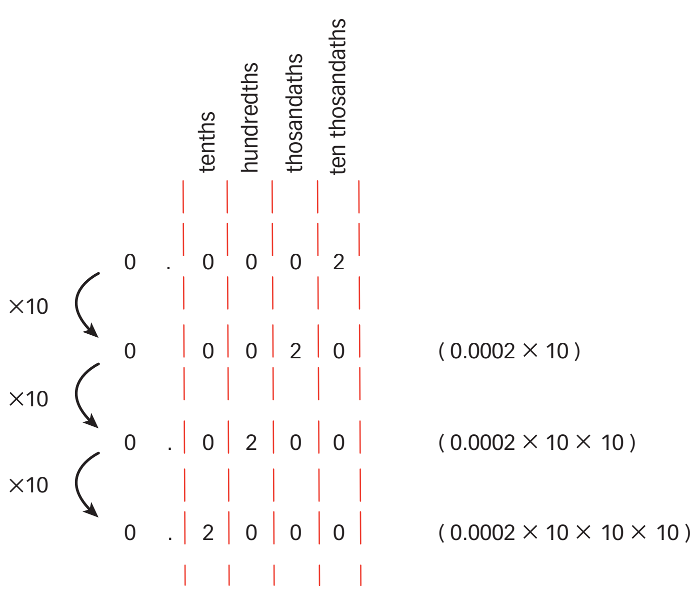

Figure 1.10 The effect of multiplying by ten

Teacher tip

Try to avoid discussing 'adding a zero' when multiplying by ten. It only clouds what is actually happening mathematically. A set of stacked paper cups can be an excellent way to show the place value of different digits in a number and start a deeper conversation about multiplying and dividing by multiples of ten (Figure 1.11).

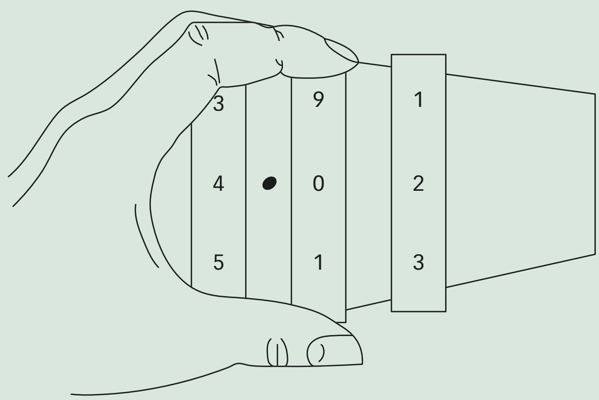

Figure 1.11 Using cups to visually demonstrate place value

Decimals are frequently used in measurements (measurement is discussed in Chapter 6), and often students are required to *round* a decimal to the nearest whole number, or tenth, or hundredth, etc. Rounding is a fairly straightforward task. One must simply determine whether a number is closer to a, or to b.

For example:

3.266 to the **nearest tenth** will either be 3.2 or 3.3. But which is it closer to? Let us use a number line to visualise it (Figure 1.12).

We can see that it is closest to 3.3, which is therefore our answer.

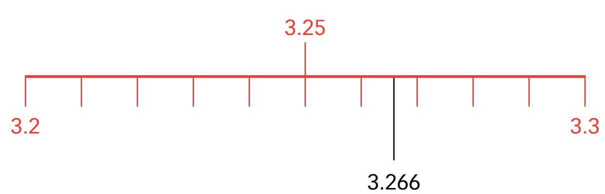

Figure 1.12 A visual demonstration of rounding to the nearest tenth

Visualising the problem on a number line makes this example easier – but what if the original number was 3.25?

This sits equidistant from 3.2 and 3.3. So rounding 3.25 to the *nearest* tenth (1 decimal place) can't be done can it? Well, we say it rounds to 3.3.

Why do we round 'up'?

Well, this is known as a tie-breaking scenario. There is no definitive 'closer' number. The general rule is to 'round up' *towards* positive infinity simply because we need convention to allow for consistency. So for example, 4.55 would round 'up' by default to 4.6 to the nearest tenth. However, it would be perfectly acceptable to round down to 4.5 if one were specifically instructed to 'round down'.

Incidentally, this causes an interesting case with *negative* numbers. Considering –4.55, we would in fact round 'up to the nearest tenth' to –4.5 by default as this is *closer to positive infinity.*

Other types of rounding

Some less common but equally valid rounding strategies are given in Table 1.7.

Table 1.7 The results of rounding when using different conventions

Decimal	Round half towards zero	Round half away from zero	Round half to even	Round half to odd
0.4	0	0	0	0
1.5	1	2	2	1
0.6	1	1	1	1
–0.5	0	–1	0	–1
–0.4	0	0	0	0

Before we move on, there is one further point to make about types of decimals (pun intended).

Rounding 0.9 to the nearest whole number will give an answer of 1. As will 0.99, and 0.999. But all of these values themselves (prior to rounding) are actually **less than 1** (see the number line in Figure 1.13).

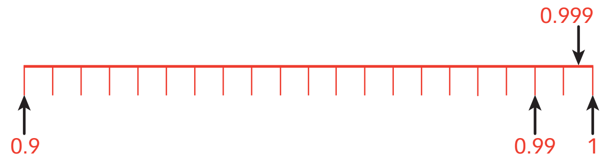

Figure 1.13 0.999 is very close to 1

However, the *recurring* decimal 0.9999 … (symbolised in mathematics as $0.\dot{9}$ is in fact **equal to 1**.

What is a recurring decimal?

A recurring decimal is the opposite of a **terminating decimal**. Recurring decimals continue infinitely, but with a recurring pattern of numbers. This could be a single digit, like with a third:

$$0.\dot{3} = 0.333333\ldots$$

Or a sequence of digits such as:

$$0.\dot{1}2\dot{3} = 0.123123123123\ldots$$

Why is $0.\dot{9}$ equal to 1?

Well, let us first consider the whole number 5. We can rewrite 5 as 5.0, or indeed 5.00 – in fact, we can write 5 as $5.\dot{0}$. And it is still just 5.

The point here is that we are simply changing the notation for the same number. This is the same as $0.\dot{9}$ and 1.

We can, of course, prove it with maths:

Let

$$m = 0.\dot{9}$$

Therefore

$$\begin{aligned} 10m &= 9.\dot{9} \\ 9m &= 10m - m = 9.\dot{9} - 0.\dot{9} = 9 \\ 9m &= 9 \\ m &= 1 \end{aligned}$$

So, somewhat counter-intuitively, the number $349.\dot{9}$ will round to 400 as the nearest 100.

Zero (part 2)

Zero has another fundamental purpose beyond being a convenient placeholder. Zero is also the gatekeeper between positive numbers and negative numbers – which in itself makes it an *even* number.

Why is zero an even number?

Take a look at the number line in Figure 1.14.

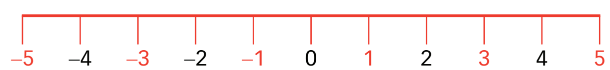

Figure 1.14 Zero is neither positive nor negative

Zero is cushioned between odd numbers, and must therefore be even. Notice that this number line also reveals another unique property of zero. It is the only number that is neither positive nor negative.

> **Teacher tip**
>
> Do not abandon number lines. They are incredibly useful for visualising numbers, estimations, place value, negatives, quantities and numerical operations. Sadly they often get dismissed as 'juvenile' later on in school. Nothing could be further from the truth.

Zero the rebellious rule breaker

Let's take a closer look at the bizarre properties of zero that made its adoption into mathematics a rather bumpy ride.

Addition and subtraction seem to stop working properly

$3 + 1 = 4$	Our answer got bigger.
$3 + (-1) = 2$	Our answer got smaller.
$3 + 0 = 3$	Addition did nothing!

Multiplication and division seem to stop working properly

$0 \times 6 = 0$

$0 \times 7 = 0$

Normally, if we multiply something by different numbers, we get different answers – but not with zero! Zero is not playing by the rules. No other numbers get these kinds of nutty results.

This particular oddity is easier to see if we switch to algebra:

$$ab = d$$

$$ac = d$$

$$\text{but } b \neq c$$

When it comes to division by zero, you may have heard about how you simply cannot do it, and it's often attributed to apocalyptic consequences. Ever wondered why?

Why is dividing by zero undefined?

It would make a degree of sense to assume that anything divided by zero equals zero. That may feel intuitive, but in fact, when we look at division as an inverse operation of multiplication, it quickly becomes apparent that dividing by zero simply doesn't work.

Take 2×3 for example:

$$2 \times 3 = 6$$

$$6 \div 3 = 2$$

By rearranging, we can see that 2 must be 6 divided by 3 by use of inverse operations. One effectively undoes the other.

This can be generalised as follows:

$$a \times b = c$$

$$c \div b = a$$

Taking this principle and applying it to division by zero would imply that, for example, $3 \div 0$ can be multiplied by some unknown number x, which will magically bring us back to 3.

In other words:

$$3 \div 0 = x$$
$$\therefore x \times 0 = 3$$

If you find a number that works for the above, do let me know! It simply doesn't work. As such, dividing by zero is not defined.

Now that we've covered counting, and zero, it makes sense to discuss counting beyond zero towards *negative infinity*.

Negative numbers

Relatively speaking, negative numbers took a long time to enter the world of maths. Counting has always had its place, and despite zero not originally being a necessity to count objects, it eventually fitted in quite nicely because of its ability to create place value and help perform written calculations. Negatives on the other hand, were simply not required. I cannot count 'negative two' loaves of bread, or subtract more physical objects than there are to begin with. At least, not practically speaking. However, negative numbers eventually found their way into maths via a concept all too familiar with many of us, *debt*. At some point most of us have happily noted our bank balance looked healthier than expected, only to be disappointed when we noticed the DR next to it. DR is a polite version of a negative symbol, i.e. you are overdrawn and you owe the bank money. The concept of negative numbers can be traced back to China and India in slightly different guises (around 200 BC and 300 AD, respectively) long before Europe adopted them. Negative numbers seem to perplex many (many!) students and it is quite understandable to see why. They do not feel intuitive, and although the concept makes a degree of sense – particularly when describing temperatures (pun intended) and bank balances. But the behaviour of negative numbers when being used in addition, subtraction, multiplication and division can start to feel a little unpredictable and, particularly for students, difficult to understand conceptually.

Adding and subtracting from a negative number

When we add or subtract a positive number *from* a negative number, the results are quite intuitive, particularly if we use a number line. For example:

$$-2 + 1 = -1$$
$$-2 - 1 = -3$$

What is going on in the sums shown above? Let's look at a simple case of adding 1 to a number, using a number line (Figure 1.15).

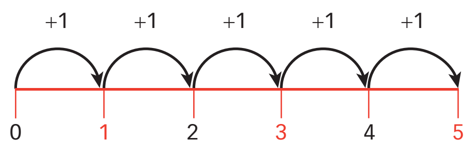

Figure 1.15 Adding 1 on a number line

Now if we extend our number line to include negative numbers, the process remains the same (Figure 1.16).

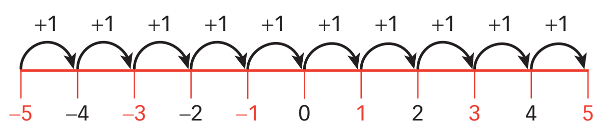

Figure 1.16 Adding 1 on a number line including negative numbers

We are moving along the number line towards infinity ('up' the number line) by adding 1. Already you may begin to see how this gets complicated for students. It's easy to start talking about the number getting 'bigger' or 'larger'. But which is 'bigger': –500 or 2? If we're talking about debt, a debt of £500 feels intuitively 'bigger' than £2 credit, especially if you're picturing the actual cash. Yet when we use inequalities (more on those later), 2 is 'greater than' –500. 'Bigger' suddenly starts to feel like a fuzzy term.

When we subtract *positive 1* from a number, the pattern of moving 'down' the number line remains the same whether starting at a point greater or less than zero (Figure 1.17).

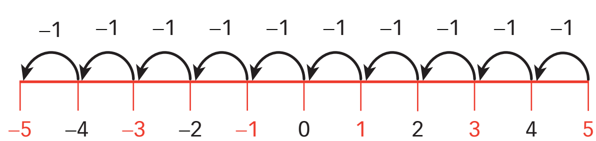

Figure 1.17 Subtracting 1 will eventually take you past zero into negative numbers

Teacher tip

Students may find it easier to conceptualise the movement along a number line if it is presented vertically, like a thermometer, rather than horizontally. This can be particularly effective when discussing changes in temperature, which is a typical way of introducing the topic.

Adding and subtracting a negative number

Things start getting a little weird when we add or subtract a *negative* number:

$$3 + (-1) = 2$$

$$3 - (-1) = 4$$

These examples begin to make more sense if we think of them as adding and subtracting *debt*. For $3 + (-1)$, I am *adding a debt* of 1. Therefore, I *owe* 1, and am left with 2. Similarly for $3 - (-1)$, I am *removing a debt* of 1. Therefore, I *am owed* 1, and I am left with 4.

Teacher tip

Another popular analogy is to consider the addition and removal of hot/cold air in a hot air balloon:

In both instances, positive numbers have a 'hot' attribute, and negative numbers have a 'cold' attribute – lending themselves towards the thermometer example mentioned earlier. Adding hot air (adding a positive number) to a balloon makes it go up (the number line), as does removing cold air (subtracting a negative). Adding cold air (adding a negative number) or removing hot air (subtracting a positive number) makes the balloon go down (the number line).

A similar example uses hot and cold water in a bath. Adding cold water makes it colder, letting out cold water makes it warmer, etc. For both metaphors, the emphasis is on trying to get students to visualise the sensibility behind the behaviours of negative numbers and move away from perceiving these properties as mystical or removed from logic.

Analogies such as debt, hot air balloons and bath water are a good way in for students to grasp the idea behind a concept, but mathematical reasoning should complement explanations. Analogies will eventually fall short, particularly as mathematics becomes more abstract.

But analogies of negative numbers as debt only take us so far. With a little mathematical thinking, the behaviour of negative numbers makes even more sense. Consider these sequences:

$$3 + 3 = 6$$
$$3 + 2 = 5$$
$$3 + 1 = 4$$
$$3 + 0 = 3$$
$$3 + (-1) = ?$$

$$3 - 3 = 0$$
$$3 - 2 = 1$$
$$3 - 1 = 2$$
$$3 - 0 = 3$$
$$3 - (-1) = ?$$

When presented in this way, it's unthinkable to consider anything other than the correct answer. It's *intuitive.*

Teacher tip

Get into the habit of referring to negative numbers as 'negative one, negative two, negative three', etc., rather than 'minus one, minus two, minus three', and similarly 'subtract' rather than 'minus'. Students can easily get confused trying to recall that 'a minus and a minus make a plus'; and this oversimplification of a complex topic does not help when presented with something like (–3)–5, which is often mistakenly assumed to be 8 or 2. Furthermore, the interchangeability of 'minus' meaning negative, and 'minus' meaning subtract is confusing for experts, let alone students just beginning to get to grips with the topic!

Not convinced? Read this out loud using only 'minus', 'three' and 'two'

$$(-3) - (-2) -2 - (-3)$$

Multiplying and dividing a negative number by a positive number

Again, when starting with negative numbers, but applying operations that involve positive numbers, the results are a little more intuitive:

$$(-3) \times 4 = -12$$

This can be thought of as four groups of –3, or more specifically:

$$(-3) \times 4 = (-3) + (-3) + (-3) + (-3) = -12$$

All that is needed here is an appreciation of integer multiplication being addition in disguise!

Division is perhaps a little trickier:

$$(-10) \div 2 = -5$$

$(-10) \div 2$ can be thought of as a debt of 10 shared equally between 2. Each share therefore must be a debt of 5 (which is represented as –5).

Multiplying and dividing a negative number by a negative number

This seems to be one of the harder things for students to master, probably because it isn't supported particularly well with any meaningful metaphor. In fact metaphors here will likely mislead students and confuse them. We will have to rely on good old mathematical reasoning, by considering the following sequence:

$$(-1) \times 3 = -3$$
$$(-1) \times 2 = -2$$
$$(-1) \times 1 = -1$$
$$(-1) \times 0 = 0$$
$$(-1) \times (-1) = ?$$

The sequence logically implies that $-1 \times -1 = 1$, which is true.

For division, which is the inverse of multiplication, it is perhaps clearer to consider rearranging the division to become multiplication.

For example, we know already that:

$$(-1) \times 1 = -1$$

Therefore, by rearranging the sum:

$$-1 \div -1 = 1$$

We have deduced a relatively abstract fact about negative numbers using a little manipulation of multiplication. It's important that time is well spent discussing these concepts in depth. Declaring the rules simply as fact without explanation is an assured way to bypass understanding of a difficult topic.

Interesting number systems around the globe

Huli

Huli (spoken in Papua New Guinea) uses a base-15 system. The number 16 translates to '15 and 1', 31 translates to '15 × 2 + 1' and so forth.

Tongan

The Tongan number system utilises repetition of the numbers 0–9. For example, 12 would be '1 2', and 20 would be '2 0'.

Georgian

The Georgian number system is in base-20. The number 30 translates to '20 and 10', 51 translates to '2 × 20 and 1 more than 10' and so forth.

Hindi

Strictly speaking, Hindi is a base-10 number system. However, in reality it acts as a base-100 system, as each number up to 100 has a unique pronunciation.

Oksapmin

Oksapmin (spoken in Papua New Guinea) has a base-27 number system, utilising 27 body parts to count from(!). In fact, the words for the numbers 1–27 are the same as the words for the body parts. Would you like a '*thumb* of carrots'? Possibly not.