last expression, you need to divide a by a sequence b_n with limit 0. But when

b_n goes to zero, $\dfrac{a}{b_n}$ could go to $+\infty$, if $b_n > 0$, or to $-\infty$, if $b_n < 0$, or not exist

at all if b_n alternates. Therefore, the expression $\dfrac{a}{0}$ must remain undefined.

Mathematicians call expressions like this "indeterminate forms." In some cases, they can be thought of as having a certain value, while in other cases, as having another value or none at all.

11. HOW LARGE IS INFINITY?

The first real opportunity where even young children can experience a "sense of wonder" in a mathematical context is when they make the basic observation that counting never ends. Typically, children are fascinated by very large numbers, and so was the 9-year-old Milton Sirotta, who became famous for inventing the name googol for the number 10^{100} in 1920. Later on, his uncle, the mathematician Edward Kasner (1878–1955), wrote about this in a book *Mathematics and the Imagination,* and still later, other imaginative young fellows named their company Google after the number googol to indicate the large amount of data handled by Internet search engines.

Although the number googol is already unimaginably large, we can easily think of even larger numbers, for example, googol $+ 1$ or 10^{googol} (which has been called googolplex), or even the factorial of googol, or googol^{googol}. The process of generating larger numbers from any given number obviously has no end. Therefore, the set of integers certainly cannot be finite. If it were, then there would be a largest number, which is impossible, because for any given number we can immediately find a bigger one just by adding 1. Therefore, the sequence of natural numbers has no end and we say that it goes to infinity, because anything that has no end or grows without limit is referred to as infinite. Thus, infinity is not a number in the usual sense—it is rather an idea or concept. And thinking about infinity is one of the most intriguing aspects of mathematics.

Already in Greek philosophy, Aristotle (384 BC–322 BC) distinguished between potential infinity and actual infinity. The process of generating larger and larger integers is an example leading to potential infinity. Potential infinity refers to an endless procedure, like counting, that can be continued indefinitely. No matter how far you go, you can still go further. This is the type of infinity encountered in calculus, when one deals with limits of sequences. We say that a sequence (α_n) goes to *infinity*, which we write symbolically as $\lim_{n \to \infty} \alpha_n = \infty$, if the sequence grows without bounds. The precise definition is the following: Whenever we assume a bound M, no matter how large, we can always find an index n_0 so that starting with this index,

all the remaining a_n are bigger than M. It is important to notice that this definition only mentions finite numbers M and n_0 and refers to a procedure that can be repeated in the same way when M is made larger and larger. This is characteristic of potential infinity—it always refers to finite quantities, which, however, may be arbitrarily large.

In contrast, actual infinity (or completed infinity) refers to mathematical situations where infinity is actually achieved. This notion has been rejected by Aristotle and other mathematicians until modern times, where, for example, Carl Friedrich Gauss (1777–1855) vehemently protested against the usage of infinity in the sense of a completed quantity. But when Georg Cantor (1845–1918) invented set theory, he also felt the need to deal with actual infinities. In set theory, actual infinity is just the number of elements (the cardinality) of infinite sets, and an infinite set can simply be defined as one that is not finite. For example, the set of all natural numbers is considered as a single, well-defined mathematical object that is given once and for all and does not change any more. It is an infinite set as has been shown before. Here infinity has been reached, as it refers to a property of a completely well-defined set. Cantor introduced the symbol $\aleph_0$ (aleph-null) to describe the (infinite) cardinality of the set of all natural numbers, and he was even able to distinguish among different types of infinity. In this context, the cardinality $\aleph_0$ of the integers is also called "countable infinity."

Actual infinity may lead to statements that appear paradoxical at first sight. Two sets are said to have the same cardinality when there exists a one-to-one mapping (a bijection) between them. For infinite sets, this notion leads to some quite amazing and unexpected results: An infinite set A will have a proper subset B that has the same cardinality as the set A. For example, the set of square integers $\{0, 1, 4, 9, 16, 25, \ldots\}$ has the same cardinality as the set $\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$ of all nonnegative integers (because there is a one-to-one correspondence between nonnegative integers and their squares, $n \leftrightarrow n^2$). Indeed, Cantor (and before him Dedekind) used this property to give an alternative definition of infinite sets: An infinite set is one that has the same cardinality as one of its proper subsets.

Another strange result is that an infinite set A has the same cardinality as the Cartesian product $A \times A = \{(x, y) | x, y \in A\}$. An example of a one-to-one mapping between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ is $(n, m) \leftrightarrow 2^n(2m+1)$, because every natural number can be written in a unique way as a product of a power of two and an odd number. Similarly, the one-dimensional line $\mathbb{R}$ contains the same number of elements as the two-dimensional plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. Of course, this could not happen with a finite set.

The counterintuitive aspects of (countable) infinite sets are paraphrased in the story of Hilbert's hotel: In a world inhabited by infinitely many people, in one of its infinitely many cities there is a hotel called Hilbert's Grand

Hotel after the famous mathematician David Hilbert (1862–1943). Hilbert's Grand Hotel has infinitely many rooms (consecutively numbered as 1, 2, 3, . . .). All rooms are occupied when a new guest arrives. The manager now has a simple idea that enables him to accommodate the new guest in the hotel. He moves the guest occupying room 1 to room 2, the guest occupying room 2 to room 3, and so on. This will leave room 1 empty for the newcomer. (Do not try to imagine the noise in the lobby, when infinitely many guests complain that they have to change room because of a single newcomer.) Actually, if the manager had told everybody to move from the room with number n to the room with number $2n$, this would even have emptied infinitely many rooms (all those with odd room numbers) and still no one would be without accommodation.

It should be noted that actual infinity is still a mathematical concept and the word actual does not imply that anything infinite exists in the real world. Although infinity is used as a mathematical concept in theoretical physics, physicists generally believe that nothing in the physical world can be infinite. For example, the number of protons in the whole universe is estimated at about 10^{80}, which is much less than a googol. On philosophical grounds, one might even argue that enormously big numbers do not exist at all except as a mathematical idea. Except for those special numbers for which we have an explicit algorithm or notation (like, for example, 10^{googol}), there is no way to write an arbitrary number that has googol decimal digits. The whole universe would not provide sufficient space, time, and material to do that, not even if we place a digit on every single elementary particle of all the galaxies in the universe. One could not represent this number in any way or compare it with others or do any computation with it. In short, there is nothing in reality that corresponds to such a number. And in mathematics, even these large numbers are infinitely tiny against infinity.

12. IS THERE ANYTHING LARGER THAN INFINITY?

The mathematical concept of infinity is tricky, but interesting. In calculus, the symbol ∞ is defined as something that is bigger than any real number, hence we have $a < \infty$ for all real numbers a. In that sense, no number is larger than infinity. But wait, mathematics is full of surprises!

In number theory, we encounter the concept of infinity as the number of elements, or cardinality, of an infinite set. An example of an infinite set is the set of all natural numbers, $\mathbb{N} = \{1, 2, 3, \ . \ . \ . \ \}$, whose cardinality is denoted by $\aleph_0$ (pronounced "aleph-null," where aleph is the first letter in the Hebrew alphabet). In fact, most of the sets that are of interest in mathematics are infinite: the set of rational numbers, the set of real numbers, the set of points in a plane, and so on.