Numbers and Operations 1

A high-quality instructional program in Grades K–8 will enable students to:

- Understand numbers, ways of representing numbers, relationships among numbers, and number systems;
- Understand the meanings of operations and how they relate to one another; and
- Compute fluently and make reasonable estimates.

OVERVIEW

The introductory discussion piece, "A Case for 'Why?'" is a manifestation of yours truly's shared constructivist philosophy of teaching math. In plain English, rather than spend all of my time teaching How To (note I said "all" and not "some"—every student needs some amount of time developing a fundamental skill set), I "invest" time in teaching Why.

But why invest in Why? I can think of three good reasons:

1. Why gets to the reasoning behind the concept much faster than How To.
2. How To becomes much easier to teach when Why is explained and reinforced—operations start to "make sense."
3. Students tend to retain the concepts longer, which lessens the amount of time necessary to reteach, reteach, reteach. . . .

So, the piece begins with an example of multiplying two mixed numbers. However, the How To is postponed through a series of visual Why steps: What does a proper fraction mean? What does multiplying two proper fractions look like? What does reducing a fraction both mean and look like? What does a

"GCF and LCM" is printed with permission from Paul Arganoff, Mathematics Department Chair, St. Francis Middle School, St. Francis, MN.

"Make It With an Abacus" is printed with permission from Ronald Greaves.

mixed number look like? What does multiplying two mixed numbers (and the resulting equivalent improper fraction) look like?

Only then is How To demonstrated, which then should make more sense than simply performing mindless arithmetic steps without the rationale. An original TI-73 graphing calculator program application (**MULTFRAX**) follows as a means for students to check their written work under adult supervision.

The rest of the chapter contains three activities: "Double Take" uses a standard deck of cards to strengthen arithmetic skills between students, "GCF and LCM" takes a fresh look at the similarities and differences between these two familiar processes, and "Make It With an Abacus" not only helps the student construct but also use a pony-bead abacus with some truly astounding insights into both estimation and precise arithmetic.

A CASE FOR "WHY?"

$$1\tfrac{1}{2} \times 2\tfrac{1}{3} = ?$$

Present this problem to students with the knowledge of how to solve it and access to today's pervasive technology, and the majority of them would first ask, "Where's my calculator?" Then, with TI-73 Explorer graphing calculators in hand, those students would engage the following keys:

1 UNIT 1 (ArrowSouth) 2 (ArrowEast) × 2 UNIT 1 (ArrowSouth) 3 ENTER

And, should the improper fraction $\frac{7}{2}$ need to be changed into mixed number form, they would press the button $\mathbf{A\frac{b}{c} \leftrightarrow \frac{d}{e}}$ to reveal $3\frac{1}{2}$.

No doubt the answer is absolutely correct, but what is at stake here? If this and similar mechanical tasks are all that is required for students to demonstrate their "proficiency" in meeting some state-level assessment standard, then we shouldn't be surprised when our country's math students continue to receive mediocre scores on international comparison tests.

Let me hasten to add that yours truly took full advantage of numerous opportunities to apply available technology in my own math classes (to *check* answers, for example). In fact, this book itself features several original programs written for the TI-73.

Moreover, the whole point of this chapter is to discuss how to get the right answer. Agreed! But, if you as the instructor are given time with your students to actually *teach* them, then it is in everyone's best long-term interest that less time be spent on *how* and more time be spent on *why* math is the way it is.

The reader is encouraged to remember four very basic teaching criteria:

> Rule 1. Since learning math is an active and accumulative process, it is far more engaging for students to "do" math than just receive it.
>
> Rule 2. Math should *always* make sense. (If it doesn't, a student's ensuing confusion will eventually lead to frustration and a loss of confidence.)
>
> Rule 3. The wider your teaching repertoire, the more students you'll have a chance to reach (aka the "bigger net" theory).
>
> Rule 4. Standing in front of a classroom and reading aloud from the textbook is still reading. Reading is generally *not* teaching (see Rule 1).

So, for our purposes here, what is the thought process that fills in all the "middle stuff"—that is, *why* did we get $3\frac{1}{2}$ as the answer to our original question? Well, there are some who would get out paper and pencil, change the original mixed numbers to improper fractions, and respond this way:

$$\frac{3}{2}\times\frac{7}{3}=\frac{21}{6}=3\frac{3}{6}=3\frac{1}{2}$$

Still others would pursue more of a "short-cut" strategy:

$$\frac{\cancel{3}}{2}\times\frac{7}{\cancel{3}}=\frac{7}{2}=3\frac{1}{2}$$

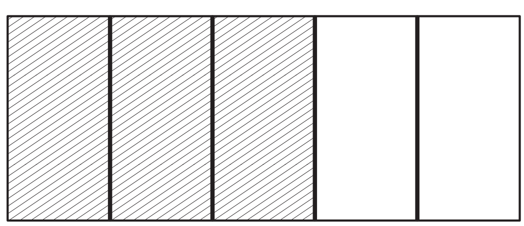
But the two processes above are strictly algorithmic—as one's textbook might explain, perform step A, then step B, and so forth. So up to now, we have identified *how* to get to $3\frac{1}{2}$ in three ways, but we have yet to determine *why*. We might successfully answer a similar state-level test question, but more than likely we haven't the foggiest notion what's actually going on here.

The explanation that follows takes into account the fact that, contrary to the discrete nature of the chapters of this book, the five paralleling NCTM standards work in conjunction with each other. The search for Why will require us to integrate the more abstract process standards (problem solving, reasoning, communication, connections, and representation).

For example, here we shall revisit the meaning of a fraction (communication) by making some visual models (representation), starting with some easier examples (a problem-solving strategy). When we introduce operations and begin to compare our results (connections), we shall be on our way (reasoning) to understanding.

That understanding should help us explain *why* the steps we took in our earlier algorithmic processes actually worked—*not just how* they worked.

1a. What does $\frac{3}{5}$ mean?



Always think "Bottoms Up!" Taking a whole unit (here we're using a rectangle), first cut it into five fifths (the bottom number, or denominator). Then color in three of those five regions (the top number, or numerator).

But a lot more math than that can be squeezed out of this one model. If, for example, we consider the rectangle to represent $1.00 US, then each of the five regions would represent $\frac{1}{5}$ of that dollar, or $0.20. And, by extension, since the word "percent" literally means "out of 100," each of the 20 cents would be equivalent to 20% of that one dollar (and 20% × 5 = 100%).

So, the three shaded regions—each of which separately equals $0.20—would altogether represent 3 × $0.20, or = $0.60. What could we do with 60¢?

1b. If I had a friend, I might split my \$0.60 "right down the middle," give him half, and keep half for myself. How much money would I now have (Figure 1.1)?

Think about this! Working with money should present little or no problem—half of \$0.60 is \$0.30 (30¢ for me, 30¢ for my friend). However, students tend to lose their insight when it comes to the same problem—in *fraction form*!

Basically, the words "half of" translate mathematically into "$\frac{1}{2}$ times," or $\frac{1}{2}\times$. So, if we rewrite 0.60 as $\frac{3}{5}$ (of a dollar), then the question becomes:

1c. What does $\frac{1}{2}\times\frac{3}{5}$ look like? (See Figure 1.2 for a model of this operation.)

Thinking "Bottoms Up!," the fraction $\frac{1}{2}$ means two regions (established by the horizontal line), of which we're only interested in one region (the top half shaded with slanted lines in the opposite direction).

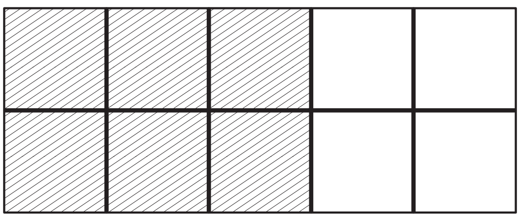


Figure 1.1

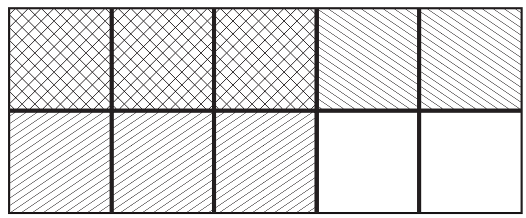


Figure 1.2

Although the answer is much more apparent for students who use two colors (one for $\frac{3}{5}$, the other for $\frac{1}{2}$) the ten regions so displayed contain only three regions that have been colored twice (or, here, have a double-slanted "mesh" design) $=\frac{3}{10}$.

Now the algorithmic process that produces the same answer—the product of the two numerators ($1 \times 3 = 3$) and the product of the two denominators ($2 \times 5 = 10$)—should begin to make more sense:

$$\frac{1}{2}\times\frac{3}{5}=\frac{3}{10}$$

2a. What is $\frac{2}{3}\times\frac{4}{5}$? (Various models related to this question will appear below.) Let's work backwards (another problem-solving strategy), beginning with creating a visual representation of the fraction $\frac{4}{5}$.

Incidentally, the reader may now wish to engage the entire class in the modeling process by passing out to each student 3-by-5 cards, two markers of different colors, a pen or pencil, and a ruler.

(Remember: "Bottoms Up!") Have each student draw four vertical lines one inch apart in order to establish the five regions that make up the denominator "5." Then, taking one of the markers, color in four of the regions to represent the numerator "4" (see Figure 1.3).

Continuing to work and read backwards, next have each student draw two horizontal lines one inch apart in order to establish the three regions that make up the denominator "3" of the other fraction in the problem (Figure 1.4).

Then, taking the other marker, color in one of the horizontal regions to represent the numerator "1" (Figure 1.5). If the question had read, "What is $\frac{1}{3}\times\frac{4}{5}$?" the answer would be the four squares that had been colored twice (or, here, have a double-slanted "mesh" design) $=\frac{4}{15}$.

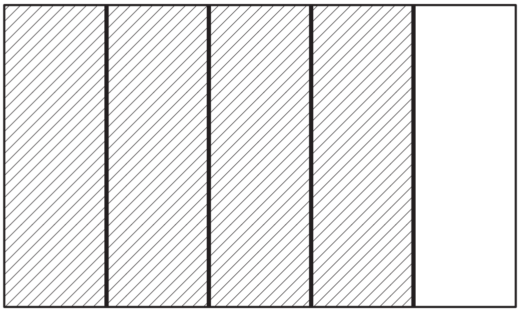


Figure 1.3

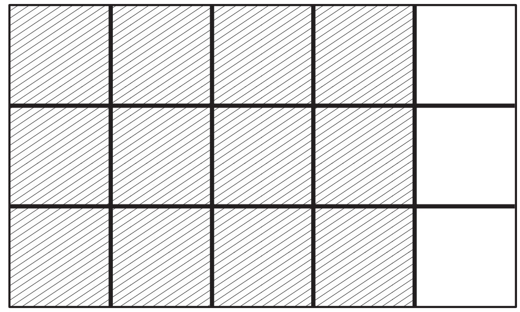


Figure 1.4

We are interested here, however, in modeling and evaluating the question, "What is $\frac{2}{3} \times \frac{4}{5}$?" So, we must color a second horizontal strip as illustrated (Figure 1.6), which means that eight of the fifteen squares are double-colored (or "meshed") $= \frac{8}{15}$.

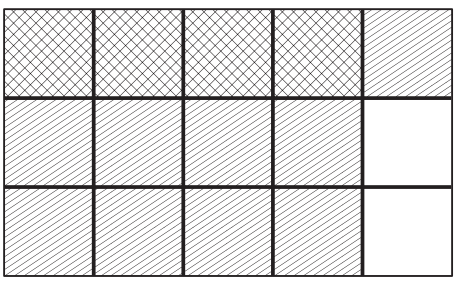


Figure 1.5

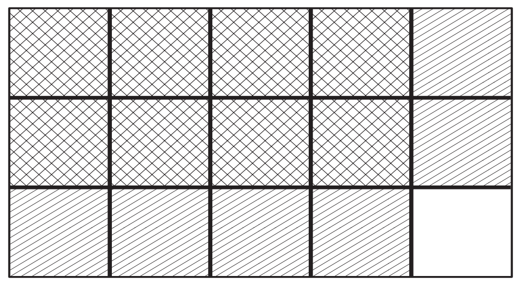


Figure 1.6

The same effect, of course, can be created with the product of the numerators ($2 \times 4 = 8$) and the product of the denominators ($3 \times 5 = 15$). But, let's keep an eye on a developing pattern (another problem-solving strategy):

$$\frac{2}{3}\times\frac{4}{5}= 2 \times \left(\frac{1}{3} \times\frac{4}{5}\right) = 2 \times \left(\frac{4}{15}\right)=\frac{8}{15}$$

2b. It may appear redundant, but what would $\frac{3}{3} \times \frac{4}{5}$ look like?

Continue by erasing the color from any single-colored square in Figure 1.7 and removing the two horizontal lines (Figure 1.8). Now, doesn't this model look familiar? How would you describe (communicate) what fraction this model represents?

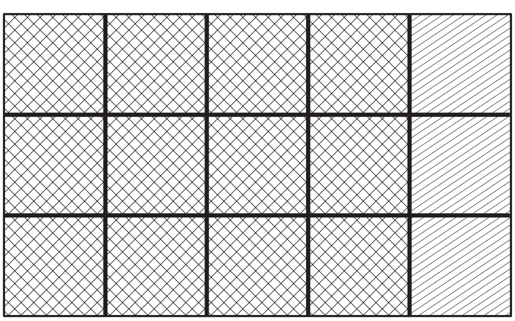


Figure 1.7

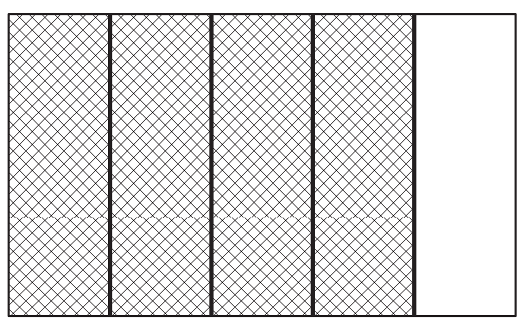


Figure 1.8

It would appear that, without bothering to simplify the fraction the more traditional way (in this case, dividing both the numerator and the denominator by 3), $\frac{12}{15} = \frac{4}{5}$. Actually, the visual concept of "simplifying" is akin to the removal of the two horizontal lines in the model!

So, if you know (can you see the model in your head?) that $\frac{3}{3} = 1$, then, because anything times 1 is itself (**multiplicative identity**) (Figure 1.9):

$$\frac{3}{3}\times\frac{4}{5}=3\times\left(\frac{1}{3}\times\frac{4}{5}\right)=3\times\left(\frac{4}{15}\right)=\frac{12}{15}$$

$$1\times\frac{4}{5}=\frac{4}{5}$$

Figure 1.9

3. So, what will $1\frac{1}{2} \times 2\frac{1}{3}$ look like, and how will it help students gain insight? (Various models related to this question will appear below.)

Classroom participation will require a little more preparatory effort on the part of the teacher. Besides the materials mentioned before, each student should also have access to several 2-by-7 copies or cutouts of the model of $2\frac{1}{3}$ (Figure 1.10), along with a pair of safety scissors.

Figure 1.10

We start with $2\frac{1}{3}$ (working backwards as done before), and we should notice how that particular mixed number is depicted—two whole rectangles and a third of another. Our solution strategy begins by thinking of the other mixed number in the problem, $1\frac{1}{2}$, as $\frac{3}{2}$ (an improper fraction) or, more important, as "three halves" when translated into English.

So, take one copy of the model, draw one horizontal line one inch from the top and bottom (Figure 1.11).

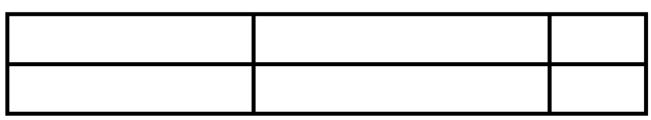


Figure 1.11

Label the six regions so created . . . :

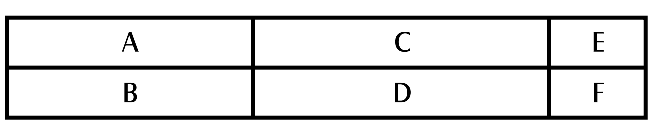


Figure 1.12

Concentrate on one of the horizontal halves (for the sake of the diagram in Figure 1.13, the top half—the "odd pieces"—are in focus). . . .

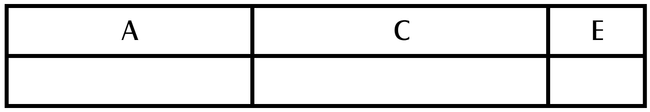


Figure 1.13

Take scissors, cut off and discard the bottom half, cut/detach the odd pieces from each other, and rearrange them into a new (rectangular) whole (Figure 1.14).

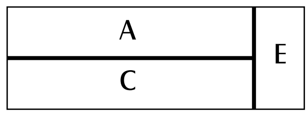


Figure 1.14

Now, regions A and C make up a new "1" (compare it with the "1's" back in the original 2-by-7 model). But the coolest aspect of this entire process is region E, because it turns out that six of our little region E's can fit vertically upon and within the "1" space (combined regions A and C) we just created.

Therefore, region E = $\frac{1}{6}$ making the new configuration equal to $1\frac{1}{6}$ or $\frac{7}{6}$.

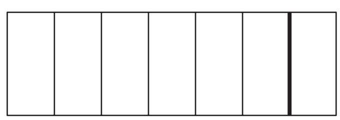


Figure 1.15

Recall the question is asking for "three halves" of $2\frac{1}{3}$, which has come to mean that we need to add three of the above configurations (three "1's" and three of the $\frac{1}{6}$ pieces).

$3\frac{3}{6} \quad = \quad 3\frac{1}{2} \quad = \quad \frac{7}{2}$

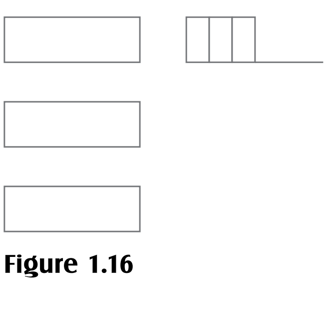


Figure 1.16

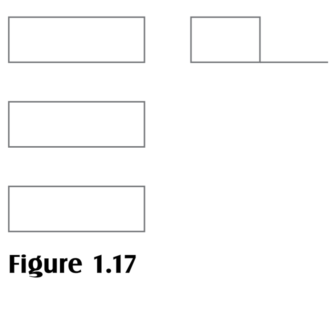


Figure 1.17

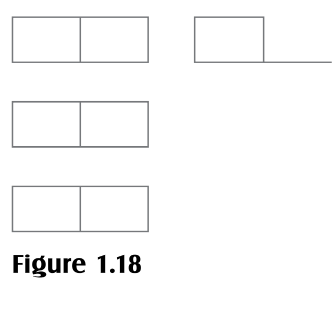


Figure 1.18

CALCULATOR APPLICATION (GRADES 4–8)

Multiplying Fractions

The original program, MULTFRAX, below will allow the user to check paper-and-pencil answers in multiplying two fractions (either proper or improper).

(Note: The "F ↔ D" portion of the second to last step of the program is the white-colored button on the TI-73's display panel that, when depressed, changes fractional answers to decimals or vice versa.)

For demonstration purposes here, we will be solving $1\frac{1}{2} \times 2\frac{1}{3}$ in improper fraction form—that is, $\frac{3}{2} \times \frac{7}{3} = ?$

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PROGRAM:MULTFRAX
:ClrScreen
:Disp "FIND_THE_
PRODUCT"
:Disp "OF_2_FRAC
TIONS"
:Disp "OF_THE_FORM"
:Disp "_"
:Disp "(A/B)_x_(C/D)"
:Pause
:ClrScreen
:Disp "FOR_(A/B),"
:Input "A_=_,"A
:Input "_and_B_=_,"B
:Disp "_"
:Disp :FOR_(C/D),"
:Input "C_=_,"C
:Input "_and_D_=_,"D
:Pause
:ClrScreen
:Disp "ANSWER_="
:Disp "_"
:Disp (AxC) / (BxD) ->
F ↔ D
:Pause
```

Input	Output
Erase Phase: Clear out all old input from your calculator ("how to" steps appear in the introduction to this book). PRGM Scroll down (ArrowSouth) to MULTFRAX	
ENTER	prgmMULTFRAX
ENTER	FIND THE PRODUCT OF 2 FRACTIONS OF THE FORM: (A/B) × (C/D)

ENTER
3 ENTER
2 ENTER
7 ENTER
3 ENTER
	FOR (A/B),
	A = 3
	and B = 2
	FOR (C/D),
	C = 7
	and D = 3

ENTER	ANSWER = 3 u $\frac{1}{2}$
ENTER ENTER (for reset) or CLEAR 2nd OFF (for shut down)	prgmMULTFRAX Done

DOUBLE TAKE

(Grades K–6) (Materials: Deck of playing cards, worksheet, pencil)

NCTM Standard: Students will be able to understand numbers, ways of representing numbers, relationships among numbers, and number systems.

There are hosts of stimulating arithmetic activities that can be produced from a standard deck of 52 playing cards. For these first three activities, students are paired off: Student A is designated as the first two-card selector, and Student B is designated the first recorder (turns are alternated).

The teacher moves around the room among the pairings, and Student A in each group takes any two cards ("Pick a card, any card. Now, do it again."). Student A then places the two cards face up, while Student B gets ready to fill in the chart (see blackline masters in appendix) with Student A's responses.

Let's suppose Student A picks the 8 of clubs and the 6 of diamonds. Student B then writes in the first column "8 clubs" in the space next to Card 1 and "6 diamonds" in the space next to Card 2.

Activity 1: Beginning Operations (Larger Number First for Grades K–3)

Whatever the two selections, the suits in this first activity are ignored. Student A uses the number values of the two cards to perform the operations of addition and subtraction (K possible?) and multiplication (his or her verbal answers are entered in the first column by Student B up to the second horizontal line in the chart). Answers: $8 + 6 = 14$, $8 - 6 = 2$, and $8 \times 6 = 48$.

When a pairing is ready to continue, Student B (turns are alternated) may select the next two cards. Student A takes on the role of recorder (now in the second column), and the activity proceeds as before.

Note: Face cards (Jack, Queen, and King) each have an equivalent value of 10. However, more able students should be challenged by giving each face card its own unique equivalence—Jack = 11, Queen = 12, and King = 13.

Activity 2: Intermediate Operations (Grades 3–6)

Whatever the two selections, the suits in this next activity are still ignored. Recall that the numerical values of the two cards first picked by Student A were 8 and 6. The task ahead is for Student A to answer all of the questions in the first column of the chart.

Since the definition of an **improper fraction** requires that the numerator be larger than the denominator, Student A rearranges the cards from side-by-side (in order of selection) to one on top of the other—in this case, $\frac{8}{6}$.

As recorded by Student B in the first column of the chart: Answers = 14, 2, and 48 (like before); improper fraction = $\frac{8}{6}$, reduced = $\frac{4}{3}$, **proper fraction** (switch the positions of the top and bottom cards) = $\frac{6}{8}$, reduced = $\frac{3}{4}$, decimal (for the proper fraction) = 0.75, percent = 75.

Activity 3: Mastery Operations (Grades 5–6)

In this activity, we begin to distinguish between "black" cards (the suits clubs and spades) and "red" cards (the suits diamonds and hearts). Moreover, black cards now represent **positive integers** and red cards negative integers.

Student A ________________________		*Student B* ________________________		
	1st Picks A	1st Picks B	2nd Picks A	2nd Picks B
Card #1:	+8			
Card #2:	−6			
Sum:	+2			
Difference:	+14			
Product:	−48			
Improper Fraction:	$-\frac{8}{6}$			
Fraction Reduced:	$-\frac{4}{3}$			
Mixed Number:	$-1\frac{1}{3}$			
Proper Fraction:	$-\frac{6}{8}$			
Fraction Reduced:	$-\frac{3}{4}$			
Decimal:	−0.75			
Percent:	−75%			

Note: There are two real-life examples of positive black and negative red—in business (when one is "in the black," there is profit) and on the leaderboards at a golf tournament.

Consider the filled-in first column of the chart on the previous page. Student B entered cards 1 and 2 not as before but as (+ 8) and (– 6), respectively. Students will also experience some new wrinkles while formulating answers in this activity because of the presence of integers, especially when subtracting and multiplying or dividing.

As with this or either of the other two activities, the teacher may choose to alter the chart depending on the ability levels of the students. The teacher may also wish to collect the worksheets at the end of the lesson in order to assess for accuracy.

Overhead Transparency Cards Activities

Have one student come to the front of the room, select two overhead transparency playing cards from a deck of yours that you have invested in and own, and display the two chosen cards on an overhead projector. Here are some alternative activities to consider for your class:

1. *Stations*: Students move around the room from the "Multiply" table to the "Decimal" table to the "Improper Fraction" table, etc., all the while writing answers to what they are observing on the overhead.

2. *Team competitions*: The room is divided in half and one student stands at the overhead. That student not only picks the cards but also spins an overhead spinner—a transparent version with sections that can be prelabeled (Add, Subtract, etc.) according to ability level. Contestants must use the two chosen playing cards and the direction from the just-spun spinner to try to guess the correct answer (the teacher keeps score and/or moderates).

3. *Buzz*: Similar to Team Competitions above in that one student chooses the cards and the operation, and the rest of the class form a line along the side of the room and answer rapid-fire questions while on their feet.

A One-On-One Activity: "Salute"

Two students face each other, and each picks one card from a face down standard deck *without* looking at their own cards (kept face down).

On the signal from the teacher, the students then raise their respective cards to their foreheads and—much like the traditional British method of palm-out saluting—show the other person their card (again, *without* ever looking at their own card).

Let's suppose Student A has the 4 of hearts (or −4) and Student B has the 5 of spades (or +5). The teacher then states an operation and a value. For example, the teacher might say, "Addition, +1."

Given first turn, Student A—seeing Student B's +5—tries to identify her *own* card and (hopefully) says either 4 of hearts or 4 of diamonds.

GCF AND LCM

Guest Contributor: Paul Agranoff

(Grades 3–8) (Materials: Worksheet, paper, and pencil)

NCTM Standard: Students will be able to understand meanings of operations and how they relate to one another.

Many students struggle with the concepts of **greatest common factor** (GCF) and **least common multiple** (LCM). When they hear of these two number theory concepts, their lack of understanding frequently causes them to confuse the two ideas and misidentify them.

The following is a different approach to understanding the difference between the GCF and LCM concepts. It also integrates other number theory ideas: prime and composite numbers, prime factorization and divisibility patterns, and common factors.

Prime and Composite Numbers

Traditionally, the technique of "decomposing" numbers is performed by using factor trees. An alternative approach to this method is prime factorization via prime number divisibility. Instruction begins with students completing the *Sieve of Eratosthenes*.

This activity "sifts out" all of the prime numbers less than 50 (or 100, if you prefer) and helps students develop a sense of prime and composite numbers. Before you begin, supply your students with a copy of the number table (found as a blackline master in appendix).

Then, complete the next six steps to determine the prime and composite numbers less than or equal to 50.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Figure 1.19

Step 1: Cross out the number 1 in the chart (Figure 1.19). The number 1 is neither **prime** (only divisible by itself and 1) nor **composite** (a product of at least two primes), but it is a factor of every number.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Figure 1.20

Step 2: Circle the number 2 (Figure 1.20). Not only is it the first prime number, but it is also the only prime that is even. Then cross out all the multiples of 2, which would all be even. (Are there patterns in the chart? If you see one, describe it.)

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Figure 1.21

Step 3: Circle the number 3 (Figure 1.21). Then cross out all the remaining multiples of 3 (single slash marks). (Are there new patterns in the chart? If you see one, describe it.) (Hint: Do you play chess?)

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Figure 1.22

Step 4: Circle the number 5 (Figure 1.22), and cross out the two remaining multiples of 5 (single slashes). (Again, are there any patterns in the chart worth mentioning?)

Step 5: Circle the number 7 (Figure 1.23)—the next prime number—and its remaining multiple (and guess what that number is before you look for it in the chart).

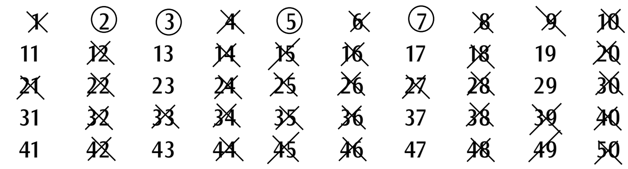

Figure 1.23

Step 6: Circle all of the remaining numbers (Figure 1.24). (These are all of the prime numbers less than or equal to 50.)

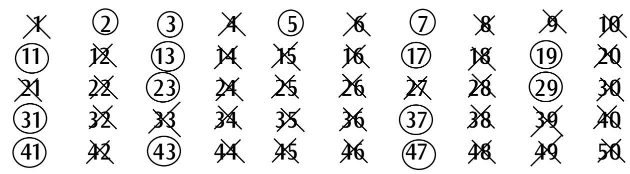

Figure 1.24

Prime Factorization by Divisibility

Rather than use factor trees, divisibility patterns (a "ladder" format) can be used to decompose composite numbers into their prime factors. For example, let's decompose the composite number 72 (Figure 1.25):

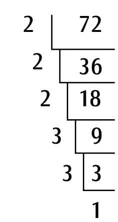

Figure 1.25

First, is 72 even? Yes, all even numbers are divisible by 2.
Is 36 even? Yes, all even numbers are divisible by 2.
Is 18 even? Yes, all even numbers are divisible by 2.
Is 9 even? No, but it is divisible by the next prime number, 3.
Is 3 even? No, but it is divisible by 3.

When we reach a value of 1, the composite number has been decomposed into its prime factors. The **prime factorization** of any number is the product of its divisors ($2 \times 2 \times 2 \times 3 \times 3 = 72$).

In fact, it turns out that composite numbers can be decomposed into their prime factors using the same technique and some basic rules of divisibility:

Divisible by 2 if the last digit of the given number is even
Divisible by 3 if the *sum* of the digits of the given number makes a new number that is divisible by 3
Divisible by 5 if the last digit of the given number is a 5 or a 0

The next prime factors (7, 11, 13, etc.) may on occasion need to be used.

Greatest Common Factor (GCF)

With the same technique as for prime factorization, two or more numbers can be decomposed simultaneously (Figure 1.26). When the prime factors that are used to decompose the two or more numbers are multiplied, the product is the GCF:

Both numbers are even: divide by 2 (common prime factor).
Both numbers are still even: repeat the process.
Two is no longer a common factor, but 3 is: divide both by 3.
There are no more common factors: process complete.

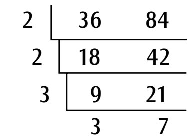

Figure 1.26

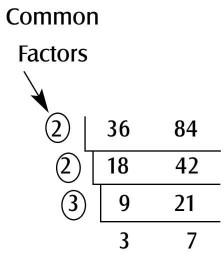

Figure 1.27

Since there are no more common factors, the numbers 3 and 7 are said to be "**relatively prime**." We can now state that the GCF of 36 and 84 is the product of the divisors/common prime factors (circled numbers in the figure below) = $2 \times 2 \times 3 = 12$.

In addition, we have modeled the decomposing of common factors used in simplifying fractions (Figure 1.27). Note that dividing both the numerator and denominator of the fraction $\frac{36}{84}$ by the GCF of 12 gives you $\frac{3}{7}$ —the simplified version of $\frac{36}{84}$ *in lowest terms*.

Least Common Multiple (LCM)

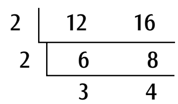

Figure 1.28

With the same technique as for the GCF, we can determine the LCM of two or more numbers. The GCF value on the left side of the process gives us the common factors. The relatively prime values at the bottom give us the uncommon prime factors. The product of the common and the uncommon factors of two or more numbers is the LCM, as shown in the following new example (Figure 1.28):

Both numbers are even: divide them by 2.
Both numbers are still even: repeat the process.
There are no more common factors: process complete.

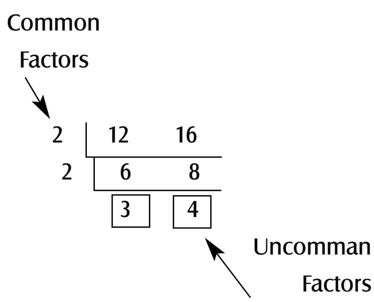

Figure 1.29

Since the numbers 3 and 4 are relatively prime, we can now state that the LCM of 12 and 16 is the product of the common factors and the uncommon factors: $2 \times 2 \times 3 \times 4 = 48$ (Figure 1.29).

(Think of an imaginary "L" going down (2×2) and then under (3×4) the entire figure. "L" stands for "Least.")

Contrasting the GCF and LCM

By using this technique after becoming more confident with finding prime numbers (the Sieve), students should recognize the difference between the GCF (the product of common factors vertically arranged on the left side) and the LCM (the product of the common *and* uncommon factors which are located along the bottom).

As you are well aware, both of these concepts are critical to a student's future success with fractional operations. The GCF is an important component in simplifying fractions, and the LCM plays a similar role with adding and subtracting fractions with unlike denominators (the LCM becomes the LCD).

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MAKE IT WITH AN ABACUS

(Grades K–5) (Materials: Listed below)

NCTM Standard: Students will be able to compute fluently and to make reasonable estimates.

This hands-on activity allows students to make their own calculating device (an abacus) and then use it to solve one- and two-digit addition, subtraction, and multiplication problems. It is the brainchild of the author's former colleague in the West Irondequoit Central School District north of Rochester, NY, retired K–12 science supervisor Ronald Greaves, and its attraction rests with the variety of tactile, sensory, and mathematical intelligence therein.

As the title of this piece suggests, there are two distinct phases to this learning activity for the student—the "assemble" phase (actually building a very inexpensive abacus), and the "calculate" phase (using that abacus to become more proficient and more confident in performing simple computations).

An **abacus** is defined as "a manual computing device consisting of a frame holding parallel rods strung with movable counters." In order for students to make their own rudimentary versions, the teacher should have the following:

1. Two prepunched tongue depressors (about 14 cm in length)
2. Five 11 cm-long dowel segments ($\frac{1}{8}$" diameter)
3. Fifty pony beads in five different colors (10 beads × 5 colors)
4. A small tube of school glue
5. A small roll of transparent tape (if necessary)
6. Some paper towels (for cleanup)

The teacher must ready the first two items on the materials list in advance, but the other four can be provisioned for the entire class. The teacher for early elementary (K and 1) grades should already have on hand several preconstructed abaci to distribute to the class.

Preparation: For a class of 25 students, the teacher will need to pre-punch at least 50 tongue depressors (2 per student—Popsicle sticks don't seem to work as well). Holes of $\frac{1}{8}$" diameter should be prepunched (reading left to right and in the middle of each stick) at these intervals: 2cm, 4.5 cm, 7 cm, 9.5 cm, and 12 cm.

The dowel segments are to be cut from $\frac{1}{8}$" × 48" craft wood dowels (available at most hardware outlets). Each 48" dowel will yield 11 segments that are 11 cm in length, which means that, since 125 segments are needed for a 25-student class (25 × 5 segments per student), at least 12 dowels need to be measured out and precut.

Cost: Items a teacher must always have are a revolving punch ($5–$10) and a pair of straight-cut snips ($10, unless he or she already has a pair for trimming garden plants).

Materials requiring constant replenishment: tongue depressors (a box of 100 sells for $5 at a specialty pharmacy), dowels (twenty-five 48" lengths sell for $5), and pony beads (a bag of 250 to 300 sells for $3 at a fabric or craft outlet. One class will easily go through 5 bags, so have some backups ready just in case).

Items such as glue, tape, and towels may already be on site.

Cost per student:
Two tongue depressors (5¢ apiece) = $0.10
Five dowel segments ($\frac{1}{2}$ dowel @ 20¢ apiece) = $0.10
Fifty beads ($\frac{1}{5}$ of a $3 bag) = $0.60
Incidentals (glue, tape, paper towels) = $0.20
Sum total cost per student = about $1.00

Steps in construction: Once each student has all of the materials together, it's time to build an abacus (about 20–30 min.).

How to Assemble

Step 1: Take all five of the dowel segments (the thinner vertical lines in Figure 1.30) and string 10 beads of the same color through each one.

Step 2: Fit each dowel segment through holes in the prepunched tongue depressors (the thicker horizontal lines), and add a small dab of glue at each juncture.

Step 3: Level off the ends of the dowel segments top and bottom (about 1.5 cm should protrude for each piece). Use paper towels for cleanup or to wipe away excess glue. The glue should take about 5 to 10 minutes to set, and then the abacus should be checked for any cracks in the tongue depressors (use a bit of tape for repairs as necessary).

How to Calculate

With apologies to any readers who have been taught otherwise, using our 10-bead abacus here requires resting it flat on a desk (Figure 1.30) and sliding all of the beads to the top tongue depressor (Figure 1.31). Those beads on the far right dowel segment are the units numbers, those to its left (second from the right) are the tens numbers, and so forth.

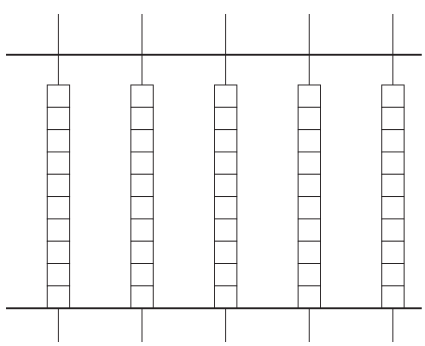

Figure 1.30

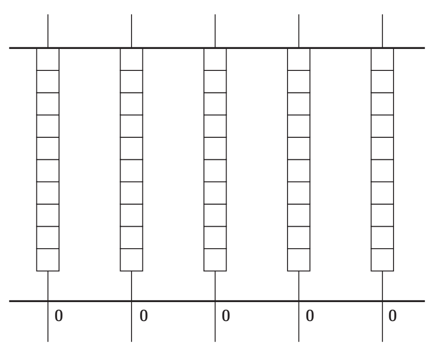

Figure 1.31

Note: Numbers have been superimposed along the bottom of the figure so that the reader may better follow the six examples that follow.

1. Simple addition: 2 + 5

Take a sharp pencil and slide the first two beads down on the units bar (as shown in Figure 1.32).

Then, from that "2 space" you just created on the units bar, count up five more beads and slide them down on top of the other two (note the new "7 space" created in Figure 1.33).

Answer: 2 + 5 = 7

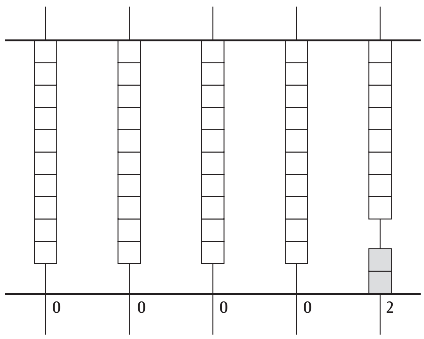

Figure 1.32

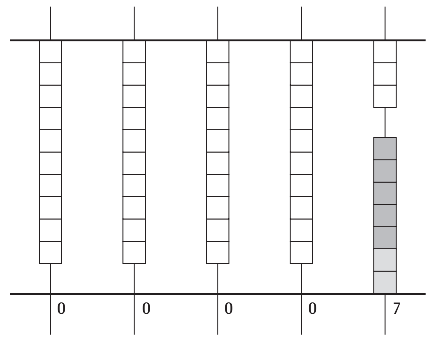

Figure 1.33

2a. Addition with "carrying" (Method 1): 8 + 9

Take a sharp pencil and slide the first eight beads down on the units bar (Figure 1.34).

If you try to slide 9 additional beads down on the units bar, you have a problem—there are only 2 left! So, slide the 2 beads down that you have available (which makes 10, as indicated in Figure 1.35).

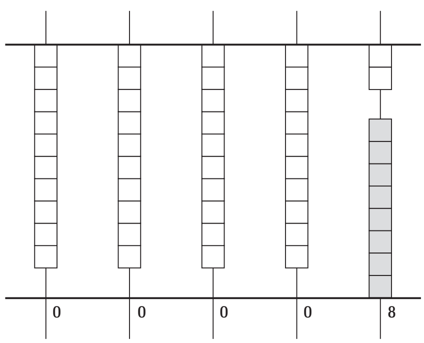

Figure 1.34

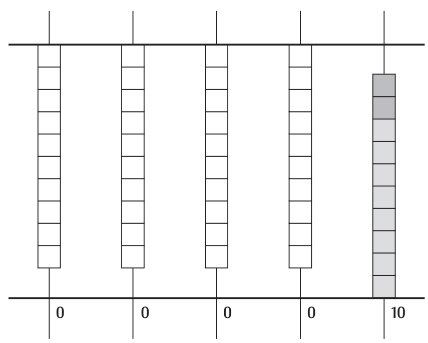

Figure 1.35

Make the equivalent version of that 10 (slide all the units beads back up and slide one tens bead down), as in Figure 1.36.

Remember that you wanted to add 9 but have already used 2 of the 9 units beads. Since 9 – 2 = 7, go back to the units bar and slide down the remaining 7 beads you need (as indicated in Figure 1.37).

Answer: 8 + 9 = 17

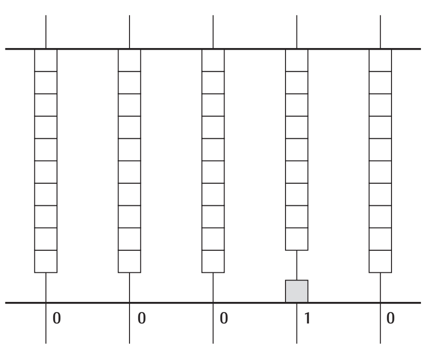

Figure 1.36

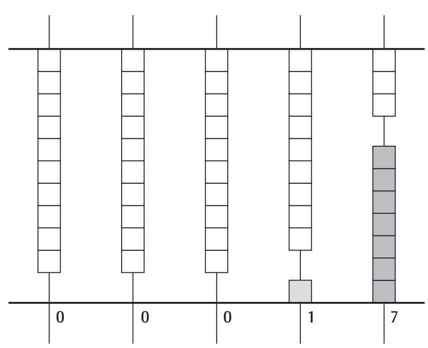

Figure 1.37

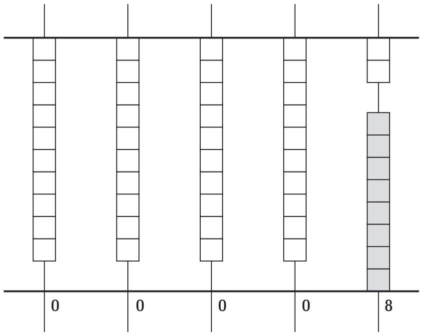

Figure 1.38

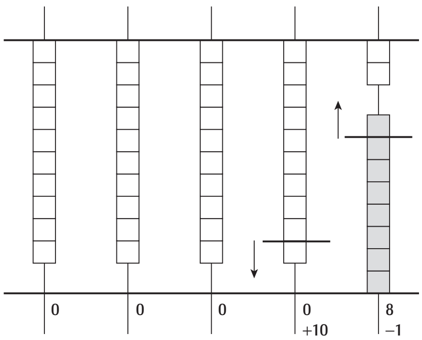

Figure 1.39

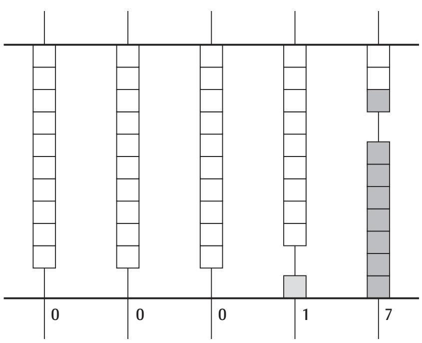

Figure 1.40

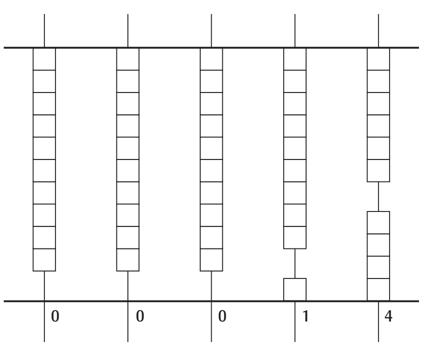

Figure 1.41

2b. Addition with "give and take" (Method 2): 8 + 9

Same problem: take a sharp pencil and slide the first eight beads down on the units bar (Figure 1.38).

At this point and for the purpose of reinforcing a key concept, the reader is directed back to part of the NCTM Standard statement at the beginning of this piece: "to compute fluently and to make reasonable estimates."

It is important for the student to understand the concept of "closeness"—in this case, *how close* is 9 to 10? If your students agree that 9 is 1 away from 10, then the class is free to discover one of the treasures of the abacus (and why it is such a great computing device for everyone).

Think of adding 9 (Figure 1.39) as actually doing two things:

"Taking" a 10 (plus) (a tens bead slides down)

and

"Giving back" a 1 (minus) (a units bead slides back up)

The result is:
$8 + 9 = 8 + (10 - 1)$
$= 8 + 10 - 1$
$= (8 + 10) - 1$
$= 18 - 1$
$= 17$ (Figure 1.40)

Author's aside: Perhaps our drive in this country to teach students the best way, only way approach to the right answer (and fortified with pencil-and-paper seatwork) may neglect an opportunity for us and for our students to apply estimation skills. This may explain in part why students in other countries tend to think more globally at a much earlier age.

3. Subtraction (no "borrowing"): 14 – 8

Take a sharp pencil and slide the first bead down on the tens bar and the first four beads down on the units bar (Figure 1.41).

Recall that we just added 9 as 10 – 1. By the "give-and-take" process, we "took" 10 (slid a tens bead down) and "gave" a 1 (slid a units bead up). If students get confused, it helps to remember this phrase: *Take Down or Give Up*.

Now we shall be subtracting as a reverse process of adding:

$$14 - 8 = 14 - (10 - 2) = 14 - 10 + 2$$

So, reversing our previous moves (see Figure 1.42), we are going to "give" 10 (slide a tens bead up) and "take" 2 (slide two units beads down).

Answer: $14 - 8 = 6$

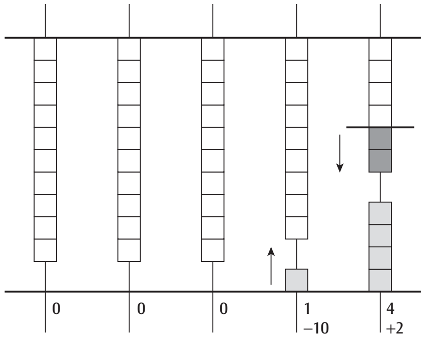

Figure 1.42

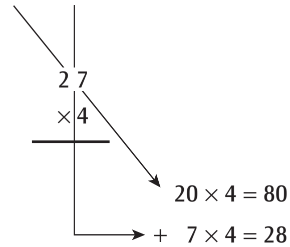

Figure 1.43

4a. Two-digit by one-digit multiplication: 27 × 4 (left to right)

Follow the arrows in Figure 1.43. First, multiply 2 × 4, which is actually 20 × 4 (think 2 × 4 with a zero tacked on the end) or 80 (Figure 1.44).

Next, multiply 7 × 4 and add the product 28 to the 80 already on the abacus (Figure 1.45).

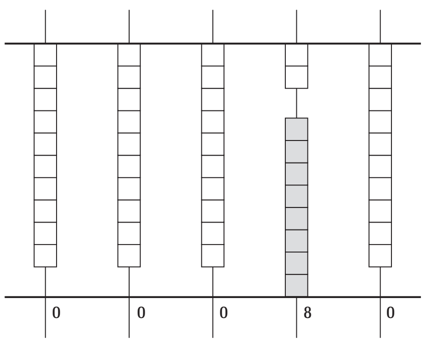

Figure 1.44

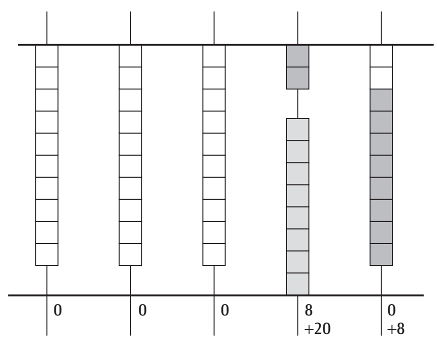

Figure 1.45

Doing so, as you can see, is going to "use up" the tens column of beads.

Just as before, we can slide all ten of a "used up" column's beads back up and slide one equivalent bead down from the column to its immediate left (Figure 1.46). In this case, we have, 10 tens = 1 hundred.

Answer: 27 × 4 = 108 (Figure 1.47)

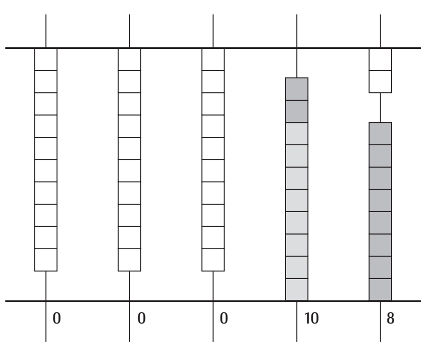

Figure 1.46

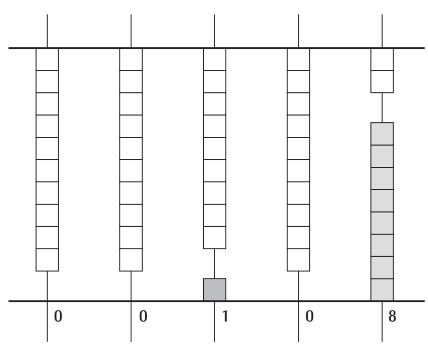

Figure 1.47

4b. Two-digit by two-digit multiplication: 59 × 36 (left to right). (Refer to Figures 1.48–1.54.)

This time, there will just be more arrows to follow in Figure 1.48 below. First, multiply 5 × 3, which is actually 50 × 30 (think 5 × 3 with two zeroes tacked on the end) or 1,500 in Figure 1.49.

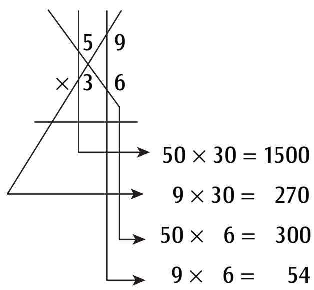

Figure 1.48

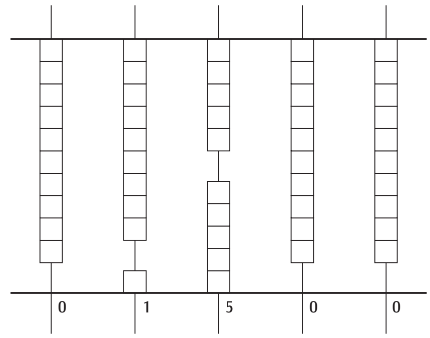

Figure 1.49

Next, multiply 9 × 3, which is actually 9 × 30 (think 9 × 3 with a zero tacked on). Add this number (the product 270) to the 1,500 already on the abacus in Figure 1.50.

In the same way, multiply 5 × 6, which is actually 50 × 6 (think 5 × 6 with a zero tacked on the end). Add this number (the product 300) to the 1,770 already on the abacus (Figure 1.51).

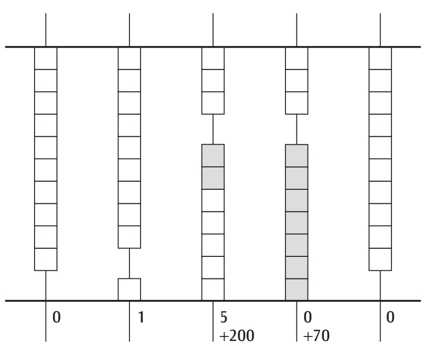

Figure 1.50

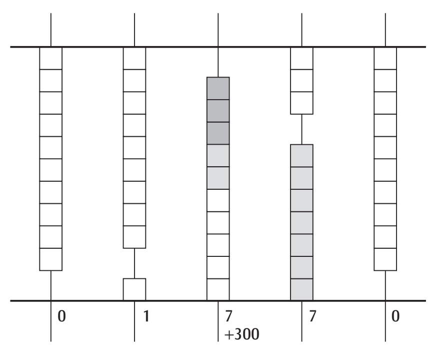

Figure 1.51

What problem do you see?

Since adding 300 will "use up" all the beads in the middle (hundreds) column, slide all ten of them back up and slide down one bead from the column (thousands) to the immediate left (as shown in Figure 1.52).

Finally, multiply 9 × 6 = 54, and add that product to the 2,070 already on the abacus.

What *new* problem do you see?

Although we'd like to add 54, we notice that only three beads are available in the tens column.

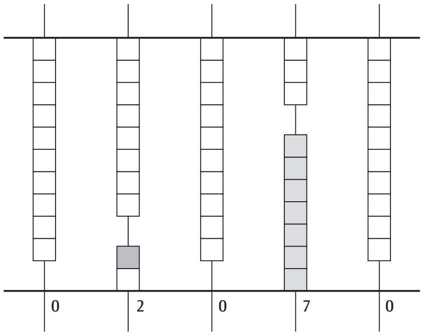

Figure 1.52

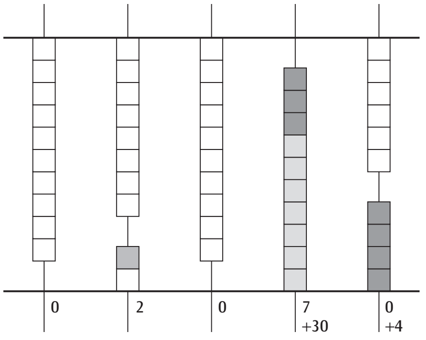

Figure 1.53

An equivalency like the one we performed in the last step won't quite do the trick here. We must slide the remaining three beads down the tens column (Figure 1.53), slide them back up (because we "used up" the column) and slide down a hundred's bead to the immediate left, and then slide two tens beads back down (because $5 - 3 = 2$).

Answer: $59 \times 36 = 2{,}124$ (Figure 1.54)

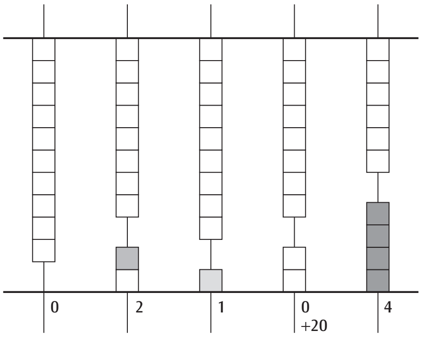

Figure 1.54

Applying the abacus in the classroom is not meant to replace any algorithmic thinking already in use. Nevertheless, it is a wonderful tool for hands-on computation and a real opportunity for students to experience place value and reverse operations (subtraction), and to review multiplication tables and multiplication as multiple addition.