# Number and Operations 1

According to the standards listed in *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM], 2000), all students in Grades 6 through 12 should be able to do the following:

- Understand numbers, ways of *representing* numbers, relationships among numbers, and number systems



- Understand meanings of operations and how they *relate* to one another



- *Compute* fluently and make reasonable estimates



The Number and Operations Standard parallels the degrees of mathematical maturity students naturally develop as they move from kindergarten through Grade 12. Students first learn what numbers are and ways to represent numbers as objects or numerals or points on a number line. Students then examine relationships among numbers, their functionality within various rule-based systems, and, finally, the invaluable assistance they render in helping to solve problems.

This chapter includes the following activities: Flexible Arithmetic, Multiplying Integers, and Areas and Mental Arithmetic. Flexible Arithmetic allows students to experience firsthand how to decompose and compose numbers while at the same time gaining a greater appreciation for a variety of standard number properties. Multiplying Integers highlights the additive identity, symmetric, distributive, and commutative properties in a unique way so that students get a better sense as to why the rules of multiplying integers are the way they are. Areas and Mental Arithmetic extends one of the notable hands-on geometry lessons from the great mathematician and mathematics educator George Pólya (1951) into a thinking strategy for students to be able to perform two-digit-by-two-digit multiplication without the use of a calculator.

# FLEXIBLE ARITHMETIC

**Grades 6–9**

This activity illustrates to students how to get answers in more than one way. The process promotes thinking mathematically about number properties. This activity correlates to the following highlighted expectation for students:

- **Understand numbers, ways of *representing* numbers, relationships among numbers, and number systems**
- Understand meanings of operations and how they *relate* to one another
- *Compute* fluently and make reasonable estimates

In the article "On Problems With Solutions Attainable in More Than One Way," Pedersen and Pólya (1984) present the idea that students who are challenged to derive the results of routine computations in different ways tend to be more creative and inquisitive in their overall approach to learning mathematics. Presented here is this author's interpretative notion of flexible arithmetic and its use as a bridge to help all students naturally move their thinking from basic operations to more abstract number properties.

Use the following example as a demonstration, and then challenge students to take any two numbers and perform a range of similar operations on their own.

Since $6 \cdot 6 = 36$, then . . .

## Distributive Property (Multiplication Over Addition): $a(b + c) = ab + ac$

How would you *rebuild* to arrive at 36?

$$6 \cdot 6 = 6 \cdot (4 + 2)$$
$$= (6 \cdot 4) + (6 \cdot 2) = 24 + 12 = 36$$

## Distributive Property (Multiplication Over Subtraction): $a(b - c) = ab - ac$

How would you *reassess* to arrive at 36?

$$6 \cdot 6 = 6 \cdot (7 - 1)$$
$$= (6 \cdot 7) - (6 \cdot 1) = 42 - 6 = 36$$

## Associative Property of Multiplication: $a(bc) = (ab)c$

How would you *regroup* to arrive at 36?

$$6 \cdot 6 = 6 \cdot (2 \cdot 3)$$
$$= (6 \cdot 2) \cdot 3 = 12 \cdot 3 = 36$$

**Multiplicative Inverse (Reciprocal): Dividing by *a* = Multiplying by $\left(\frac{1}{a}\right)$ Commutative Property of Multiplication: *ab* = *ba* and Associative Property of Multiplication: *a*(*bc*) = (*ab*)*c***

How would you *rearrange* to arrive at 36?

$$\begin{aligned} 6 \bullet 6 &= 6 \bullet \left(\frac{18}{3}\right) \\ &= 6 \bullet \left[18 \bullet \left(\frac{1}{3}\right)\right] = 6 \bullet \left[\left(\frac{1}{3}\right) \bullet 18\right] \\ &= \left[6 \bullet \left(\frac{1}{3}\right)\right] \bullet 18 = \left[\left(\frac{1}{3}\right) \text{ of } 6\right] \bullet 18 \\ &= 2 \bullet 18 = 36 \end{aligned}$$

**Distributive Property (Multiplication Over Addition): *a*(*b* + *c*) = *ab* + *ac* and Commutative Property of Multiplication: *ab* = *ba***

How would you *rewrite* to arrive at 36?

$$\begin{aligned} 6 \bullet 6 &= 6 \bullet (3.5 + 2.5) \\ &= 6 \bullet \left[\left(3 \text{ and } \frac{1}{2}\right) + \left(2 \text{ and } \frac{1}{2}\right)\right] \\ &= 6 \bullet \left[\left(3 + \frac{1}{2}\right) + \left(2 + \frac{1}{2}\right)\right] \\ &= \left[6 \bullet \left(3 + \frac{1}{2}\right)\right] + \left[6 \bullet \left(2 + \frac{1}{2}\right)\right] \\ &= \left[(6 \bullet 3) + \left(6 \bullet \frac{1}{2}\right)\right] + \left[(6 \bullet 2) + \left(6 \bullet \frac{1}{2}\right)\right] \\ &= [18 + (\text{half of } 6)] + [12 + (\text{half of } 6)] \\ &= [18 + 3] + [12 + 3] \\ &= 21 + 15 = 36 \end{aligned}$$

**Distributive Property (Multiplication Over Addition): *a*(*b* + *c*) = *ab* + *ac* or First Outside Inside Last (FOIL): (*a* + *b*) • (*c* + *d*) = *ac* + *ad* + *bc* + *bd* and Commutative Property of Multiplication: *ab* = *ba***

How would you *recall* to arrive at 36?

$$\begin{aligned} 6 \bullet 6 &= (4 + 2) \bullet (4 + 2) \\ &= [4 \bullet (4 + 2)] + [2 \bullet (4 + 2)] \\ &= [(4 \bullet 4) + (4 \bullet 2)] + [(2 \bullet 4) + (2 \bullet 2)] \\ &= [16 + 8] + [8 + 4] \\ &= 24 + 12 = 36 \end{aligned}$$

### FOIL: $(a + b) \cdot (c + d) = ac + ad + bc + bd$ and Order of Operations: Parentheses/Exponents/Multiply or Divide/Add or Subtract (PEMDAS)

How would you *respond* to arrive at 36?

$$\begin{aligned} 6 \cdot 6 &= (3 + 3) \cdot (3 + 3) \\ &= (3 \cdot 3) + (3 \cdot 3) + (3 \cdot 3) + (3 \cdot 3) \\ &= 4 \cdot (3 \cdot 3) \\ &= 4 \cdot 3^2 \\ &= 4 \cdot 9 = 36 \end{aligned}$$

After introducing the flexible arithmetic concept, step back and allow students to create their own arithmetic relationships that illustrate the various number properties exhibited above. Tailor the degree of difficulty to match students' ability levels and willingness to explore different approaches.

## CALCULATOR APPLICATION

### Multiplication Facts

```
PROGRAM:FLSHCRDS
:Lbl 99
:randInt(1,10) → A
:randInt(1,10) → B
:Disp "WHAT_IS," A
:Disp "TIMES," B
:Pause
:Disp "_"
:Disp A × B
:Disp "_"
:Pause
:ClrScreen
:Goto 99
```

The application shown in the sidebar, using the TI-73 Explorer, is an original program titled "FLSHCRDS." Like Flash Cards, it is designed to help students practice and master multiplication facts through random number-generated multiplication problems:

NOTE: For help with any aspect of writing, editing, executing, or sharing programs, please refer to the Programming chapters in the Graphing Calculator Guidebooks, which accompany the TI-73 (Chap. 12) and the TI-83/84 Plus (Chap. 16).

### Using the Application With Students

| Your Input | Output (Example) |
|---|---|
| *Press:* **ON**<br>**PRGM**<br>(*Scroll down to* **FLSHCRDS**) | |
| **ENTER** | **prgmFLSHCRDS** |
| **ENTER** | **WHAT IS**<br>**3**<br>**TIMES**<br>**4** |
| **ENTER** | **WHAT IS**<br>**3**<br>**TIMES**<br>**4**<br>**12** |

| Your Input | Output (Example) |
|---|---|
| **ENTER** | **WHAT IS**<br>**9**<br>**TIMES**<br>**7** |
| **ENTER** | **WHAT IS**<br>**9**<br>**TIMES**<br>**7**<br>**63** |
| **ENTER**<br>(*for a new question, etc.*) | |

## Instructional Ideas

1. There are a couple of reasons why two **Pause** steps were included in the "**FLSHCRDS**" program. First, for teachers working in a whole-class setting, using this program on one's overhead with a ViewScreen attachment allows for lots of questions to be posed rapid-fire for students seated around the room.

   Second, for those who wish to (or need to) work one-on-one, students can use the program and answer as many questions as circumstances warrant while practicing under direct adult supervision.

2. The **randInt(0,10)** steps were preset to generate random integers to be multiplied (or "multiplicands") between 0 and 10, inclusive. Those numbers can easily be adjusted higher or lower in the program (under "EDIT") to match student ability levels.

3. If any of your students need to reinforce their addition and/or subtraction skills, then this program can also be edited by first changing the word "TIMES" to either "PLUS" or "MINUS" and then replacing the "X" sign with either "+" or "–," respectively.
4. When you are finished and wish to escape the program, press the following:

   **2nd OFF**

   **ON**

   **CLEAR**

## MULTIPLYING INTEGERS

### Grades 6–8

This activity shows students how to develop an example to justify the rule. Try simple arithmetic as an invitation to help your students recognize patterns and then to extend their thinking into more formal mathematical statements open to further inquiry and discussion. This activity correlates to the following highlighted expectation for students:

- Understand numbers, ways of *representing numbers,* relationships among numbers, and number systems
- **Understand meanings of operations and how they *relate* to one another**
- *Compute* fluently and make reasonable estimates

In this activity, students explore a logical argument supported by combining four number properties—two new ones and two others from the previous activity:

- Additive identity property: $a + (-a) = 0$
- Symmetric property of equality: if $a = b$, then $b = a$
- Distributive property (multiplication over addition): $a(b + c) = ab + ac$
- Commutative property of multiplication: $ab = ba$

An effective instructional approach is to use the time-tested **scientific method.** First, consider Pólya's (1951) four-step interpretation of the scientific method: understand, plan, execute, and look back. A good way to help students remember those four steps is to use the mnemonic U-PLEX-L (*U*nderstand, *PL*an, *EX*ecute, *L*ook back).

Students should *understand,* for example, that (+3) • 0 = 0 and that (+2) + (–2) = 0 by the additive identity property and then that 0 = (+2) + (–2) by the symmetric property of equality.

The *plan* is to substitute (+2) + (–2) for 0. The *execution* is inherent in the distributive property phase.

The rule comes from the last, or *look back,* step. Reading vertically, note that (+3) • (+2) = (+6) but that (+3) • (–2) must be (–6), which is the whole essence of Rule 1: A positive times a negative is a negative.

### Rule 1: A Positive Times a Negative Is a Negative

Since (+3) • 0 = 0 (any nonzero integer times zero equals zero) and since (+2) + (–2) = 0, then

$$(+3) \bullet [0] = 0$$
$$(+3) \bullet [(+2) + (-2)] = 0 \text{ (substitution)}$$
$$[(+3) \bullet (+2)] + [(+3) \bullet (-2)] = 0 \text{ (distributive property)}$$
$$[(+6)] + [?] = 0$$

What must the value of ? be to get an overall result of 0?
According to the additive identity property, ? = (–6).

### Rule 2: A Negative Times a Negative Is a Positive

Since (–2) • 0 = 0 and since (+3) + (–3) = 0, then

(–2) • [0] = 0
(–2) • [(+3) + (–3)] = 0 (substitution)
[(–2) • (+3)] + [(–2) • (–3)] = 0 (distributive property)
[(+3) • (–2)] + [?] = 0 (commutative property)
[(–6)] + [?] = 0

What must the value of ? *have* to be to get an overall result of 0?
According to the additive identity property, ? = (+6).

#### One Step Further

Challenge students to uncover a signs pattern connecting both of the preceding rules. Encourage further exploration by having students use a four-function calculator for checking and by having them work through division problems.

- If the signs of both integers are the *same,* their product is *positive*.
- If the signs of both integers are *different,* their product is *negative*.

# AREAS AND MENTAL ARITHMETIC

## Grades 6–10

This activity clarifies that $a^2 - b^2 = (a + b) \bullet (a - b)$, which is the difference of two squares, and connects algebra and geometry. This activity correlates to the following highlighted expectation for students:

- Understand numbers, ways of *representing* numbers, relationships among numbers, and number systems
- Understand meanings of operations and how they *relate* to one another
- ***Compute* fluently and make reasonable estimates**

The human brain is an incredibly powerful calculator and a much faster device than a four-function calculator. To help students tap that brainpower, share with them some of the secrets that you, as their teacher, possess. The following paper-cutting activity is a good way to reveal one of those secrets.

The activities of paper folding and paper cutting have been designed to help students see abstract concepts from algebra by touching, moving, and interpreting real objects of their own design (Sobel & Maletsky, 1988). The following activity provides a glimpse into a host of engaging opportunities for students to sharpen a wide variety of their mathematical skills.

Before you begin, make sure students understand that the area of a rectangle is measured in square units. It can be found in either of two ways:

1. Counting the number of square units within the rectangle *or* (if counting a discrete number of squares is not possible)
2. Using the formula $A = \text{length} \bullet \text{width} = l \bullet w$

Each student needs the following materials:

- One piece of square paper (size does not matter)
- Scissors
- A four-function calculator
- One piece of 8½ × 11-inch paper
- A sharp pencil

## Discovering the Secret

1. Using a square piece of paper, make a diagonal crease running from the top-left corner to the bottom-right corner. With the square open, label the left and top sides *a*. See Figure 1.1.
2. Mark a point about a third of the way up the diagonal from the bottom-right corner. Draw two perpendicular lines from that point. See Figure 1.1.
3. Cut along the two perpendicular lines you drew in Step 2 to detach a square from the bottom-right corner. Label the left and top edges of the just-removed square *b*. Label the remaining two sides of the new hexagon $a - b$. See Figure 1.2.
4. Cut the hexagon along the creased diagonal to make two congruent trapezoids.
5. Flip over the left-hand trapezoid, and slide the two trapezoids together to form a rectangle. See Figure 1.3.

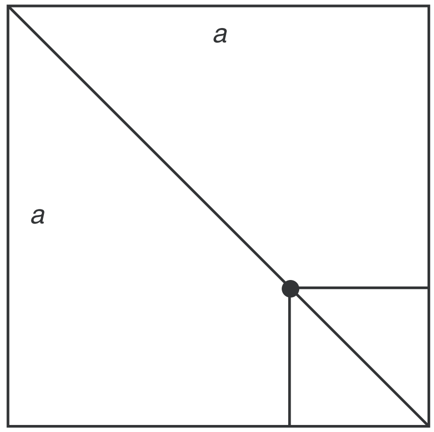


**Figure 1.1**

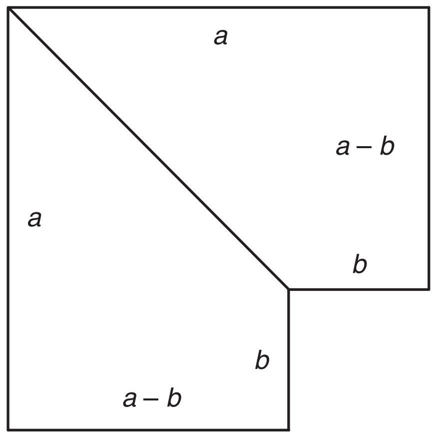


**Figure 1.2**

Once you work through the activity with students, share the following formula, or secret, with them:

If the hexagon (the original square piece of paper with the small square cut out of one corner) has area = $a^2 - b^2$

*and*

If the resulting rectangle (formed by flipping and sliding one of the two congruent trapezoids) has area = $(a + b) \bullet (a - b)$

*and*

If the hexagon and the rectangle both have the same area

Then $a^2 - b^2 = (a + b) \bullet (a - b)$

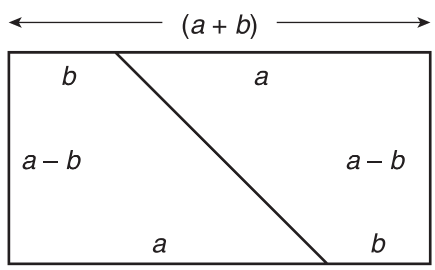


**Figure 1.3**

An algebra student might recognize that formula as having resulted from factoring (the difference of two squares), but note how the formula is simply a summary of moving pieces of paper a certain way to make geometric figures and study their areas.

## Performing Mental Arithmetic

With the paper-cutting process now complete, focus students' attention on how they can perform mental arithmetic (specifically, multiplying two-digit numbers by two-digit numbers) much more quickly than they can with a calculator. As a prerequisite exercise, challenge your students to identify all **perfect square** numbers from 1 to 625, inclusive. These numbers constitute one of the easiest and most prevalent groups of natural numbers for students to remember. These numbers continuously reappear throughout students' entire mathematical experience—from multiplication tables to area formulas to the Pythagorean Theorem to the Quadratic Formula and beyond.

Any number that is the product of the same two whole-number factors is a perfect square. Those numbers of 1 to 625, inclusive, are as follows:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 16 | 49 | 100 | 169 | 256 | 361 | 484 |
| 4 | 25 | 64 | 121 | 196 | 289 | 400 | 529 |
| 9 | 36 | 81 | 144 | 225 | 324 | 441 | 576 |
| | | | | | | | 625 |

Now challenge students to solve the equation $22 \cdot 18 = ?$ by following these three steps:

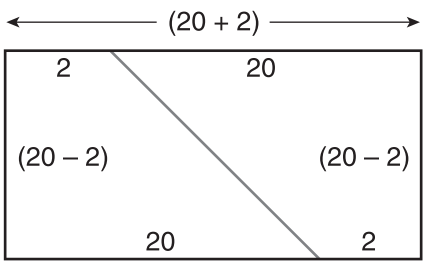


**Figure 1.4**

### *Step 1*

Compute the **mean** (average) of 22 and 18. It may appear difficult at first, but the quickest and most efficient way to do this is to identify the middle number between 22 and 18, which is 20.

### *Step 2*

Determine how far apart either 22 is from 20 *or* 20 is from 18. That "space" either way is 2.

### *Step 3*

Try to imagine how the rectangle and the hexagon would look with the numbers 20 and 2 in place of the letters *a* and *b*, respectively. See Figures 1.4 and 1.5.

$$22 \cdot 18 = (20 + 2) \cdot (20 - 2)$$

Rearrange the rectangle to look like the hexagon again:

$$(20 + 2) \cdot (20 - 2) = 20^2 - 2^2$$

(area of big square minus little square)

$$= 400 - 4$$
$$= 396$$

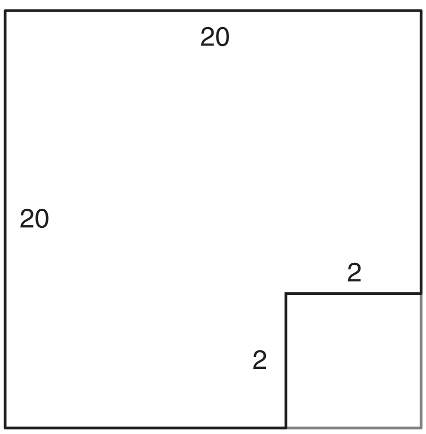


**Figure 1.5**

## CALCULATOR APPLICATION

### Perfect Squares

```
PROGRAM:PERFSQRS
:ClrScreen
:Disp "THE_FIRST_25"
:Disp "PERF._SQUARES"
:For (A, 1, 25, 1)
:Disp A²
:Pause
:End
```

Any number that is the product of the same two whole-number factors is referred to as a "perfect square." The subset of perfect squares from 1 to 625, inclusive, can be generated with the application at the right, an original program titled "**PERFSQRS**":

NOTE: The command "ClrScreen" for TI-73 users is the same as the command "ClrHome" for TI-83/84 Plus users.

### Using the Application With Students

| Your Input | Output |
|---|---|
| *Press:* **ON**<br>**PRGM**<br>(*Scroll down to* **PERFSQRS**) | |
| **ENTER** | **prgmPERFSQRS** |
| **ENTER** | **THE FIRST 25**<br>**PERF. SQUARES**<br>**1** |
| **ENTER** | **THE FIRST 25**<br>**PERF. SQUARES**<br>**1**<br>**4** |
| **ENTER** | **THE FIRST 25**<br>**PERF. SQUARES**<br>**1**<br>**4**<br>**9** |
| **ENTER**<br>*(for the next perfect square, etc.)* | **16**<br>**25**<br>**36**<br>**49**<br>**64** |

*(Continued)*

(Continued)

| Your Input | Output |
|---|---|
| | **81**<br>**100**<br>**121**<br>**144**<br>**169**<br>**196**<br>**225**<br>**256**<br>**289**<br>**324**<br>**361**<br>**400**<br>**441**<br>**484**<br>**529**<br>**576**<br>**625** |
| Pressing **ENTER** after reading the last number in the list (625) yields: | **prgmPERFSQRS**<br>**Done** |

## Instructional Ideas

1. The **Pause** step included in the "**PERFSQRS**" program lets teachers working in a whole-class setting use this program on one's overhead with a ViewScreen attachment for either recitation or call-upon purposes.

   Students may not realize the importance of becoming very familiar with these numbers. But many situations exist—simplifying radicals, Pythagorean Theorem, and Quadratic Formula exercises, to name three—in which recognizing perfect squares makes the problem at hand much easier to solve.

2. An interesting sidelight to this list involves a sequence of increasing odd numbers. Note that each successive perfect square is the next odd-number difference away from its immediate predecessor. Starting from zero, 0 **+ 1** = 1, 1 **+ 3** = 4, 4 **+ 5** = 9, 9 **+ 7** = 16, 16 **+ 9** = 25, and so on.

   For a more visual explanation, please refer to the building of squares with two-color counting tiles in the "Visual Thinking" unit of Chapter 7.

3. As was the case with the "**FLSHCRDS**" program, the number 25 in this program can be adjusted according to student ability levels.

4. When you receive the **Done** statement and wish to escape,

   *Press:* **CLEAR**

   However, if you desire an immediate replay after being told you are done,

   *Press:* **ENTER**

## Let's Review

**Step 1:** Find the middle number between the two given *multiplicands* (the two numbers being multiplied). This number is the same as the larger number in the paper hexagon.

**Step 2:** Count how far away that middle number is from either of the two multiplicands. This number is the same as the smaller number in the paper hexagon (one side of the cut-out square).

**Step 3:** Square the larger number (from Step 1) and subtract from it the square of the smaller number (from Step 2) to get the answer.

The following are some additional equations that you can use to challenge your students:

$18 \bullet 12 = ?$

**Answer:** $18 \bullet 12 = (15 + 3) \bullet (15 - 3) = 15^2 - 3^2 = 225 - 9 = 216$

$23 \bullet 15 = ?$

**Answer:** $23 \bullet 15 = (19 + 4) \bullet (19 - 4) = 19^2 - 4^2 = 361 - 16 = 345$

## One Step Further

Test your students' estimation skills. Have them show that the number 9,991 is composite (that is, *not* prime).

Have students estimate and then visualize backward (recall "Discovering the Secret") and think from the hexagon to the rectangle.

$$9{,}991 = 10{,}000 - 9 = 100^2 - 3^2 = (100 + 3) \bullet (100 - 3) = 103 \bullet 97$$
$$a^2 - b^2 = (a + b) \bullet (a - b)$$